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NON-COMMUTATIVE POLYNOMIALS AND CYCLIC ALGEBRAS[1]

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Introduction

Consider the cyclic algebra $\mathfrak{F}'(\Pi)$ whose basis consists of the products $a^j x^k$ ($j, k = 0, 1, \cdots, r-1$) over a field $\mathfrak{F}$ where a generates a cyclic field $\mathfrak{F}' = \mathfrak{F}(a)$ over $\mathfrak{F}$ and

$$xa = a^{(1)}x \tag{1}$$

$$\Pi = x^r - \gamma = 0 \qquad (\gamma \neq 0) \tag{2}$$

where a is an element of $\mathfrak{F}'$ such that the correspondence a into $a^{(1)}$ generates the Galois group of $\mathfrak{F}'$ over $\mathfrak{F}$. We consider $\mathfrak{F}'(\Pi)$ as a residue algebra of the polynomial Π in the domain $\mathfrak{F}'[x]$ of all polynomials in x whose coefficients $\subset \mathfrak{F}'$ and for which multiplication is defined by (1). The main purpose of the present paper is to show that the problem of determining the division algebra part of $\mathfrak{F}'(\Pi)$ in the Wedderburn decomposition may be reduced to the problem of finding a single irreducible factor of Π in $\mathfrak{F}'[x]$. In §1 we discuss the factorization theory in $\mathfrak{F}'[x]$. In §2 and §3 we determine the relation between $\mathfrak{F}'(\Pi)$ and an algebra associated with a factor of Π. Finally, in §4 we derive a "norm" condition in order that Π have an irreducible factor of degree t and from this condition the usual norm conditions for cyclic algebras follow as simple corollaries. The methods used are applicable to the algebras defined by (1) and (2) where $\mathfrak{F}'$ is any division algebra of finite order over $\mathfrak{F}$. We hope to consider this case in a later paper.

In the course of the preparation of this paper, I had the privilege of discussing its details with Professor Wedderburn. I am very grateful to him for the stimulus of these discussions.

1. Cyclic Polynomial Domains

Let $\mathfrak{F}' = \mathfrak{F}(a)$ be a cyclic field of order r generated by a over the field $\mathfrak{F}$ and $a^{(1)} = \theta(a)$ be another root of the minimal equation satisfied by a in $\mathfrak{F}$ such that the correspondence a into $a^{(1)}$ is a generating automorphism of the cyclic Galois group of $\mathfrak{F}'$ over $\mathfrak{F}$. We define the *cyclic polynomial domain*, $\mathfrak{F}'[x]$, as the set of expressions of the form

$$P = a_0 + a_1 x + \cdots + a_n x^n$$

[1] Presented to the American Mathematical Society, June 19, 1933.

where the $a_i \subset \mathfrak{F}'$ and x is an indeterminate. Addition of such expressions is defined in the usual manner. Multiplication is defined by (1) together with the distributive and associative laws. From (1) one obtains

$$x^k b = b^{(k)} x^k$$

where $b = b(a)$ is a polynomial of degree $< r$ in a and $b^{(k)} = b(\theta^{(k)}(a))$, $\theta^{(k)}$ being the k-th iterative of θ. It is easily verified that $\mathfrak{F}'[x]$ is a domain of integrity. It may also be considered as an algebra with an infinite basis over $\mathfrak{F}$. In the present section we take the former viewpoint and give an outline of the theory of factorization in $\mathfrak{F}'[x]$. We mention first certain well known properties of $\mathfrak{F}'[x]$ referring to the papers of Wedderburn and of Ore for the proofs.[2]

For any two polynomials P and Q there exists a division process leading to polynomials S, R and S_1, R_1 such that

$$P = SQ + R \qquad P = QS_1 + R_1$$

where the degree of R, $\delta(R) < \delta(Q)$ and $\delta(R_1) < \delta(Q)$. Further, S, R and S_1, R_1 are unique.

The existence of a highest common right (left) factor of P and Q, $(P, Q)_R$ ($(P, Q)_L$) and a least common left (right) multiple, $[P, Q]_L$ ($[P, Q]_R$) follows from the division process.[3] These are unique except for left (right) unit multipliers. We have

$$\delta(PQ) = \delta[P, Q]_L + \delta(P, Q)_R = \delta[P, Q]_R + \delta(P, Q)_L. \tag{3}$$

There exists A_1, B_1 and A_2, $B_2 \subset \mathfrak{F}'[x]$ such that

$$(P, Q)_R = A_1P + B_1Q \qquad (P, Q)_L = PA_2 + QB_2.$$

We recall the definition of a *generalized left-transform* of P by Q as $P^* = [P, Q]_L Q^{-1}$.[4] If $(P, Q)_R = 1$, then (3) shows that $\delta(P) = \delta(P^*)$ and P^* is said to be *left-equivalent*[5] to P. Right-transforms and right-equivalence may be defined in an analogous way. We refer to Ore's paper (O. pp. 488–491) for a discussion of transforms, recalling merely that right- and left-equivalence imply each other and hence we drop the modifiers "right" and "left" in what follows.

The fundamental factorization theorem for $\mathfrak{F}'[x]$ is the following

THEOREM 1. *A polynomial A may be expressed as $P_1P_2 \cdots P_s$ where the P's are irreducible. If $Q_1Q_2 \cdots Q_t$ is another factorization of A into irreducible factors, then $s = t$ and the P's and Q's are equivalent in pairs.*

We refer to Ore's paper (O. p. 494) for the proof.

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[2] J. H. M. Wedderburn, On continued fractions in non-commutative quantities, Ann. of Math., **15**, (1913–1914), pp. 101–105 and Non-commutative domains of integrity, Jour. für Math., **167**, (1932), pp. 129–141. The latter paper will be referred to as W. Ö. Ore, Theory of non-commutative polynomials, Ann. of Math., **34**, (1933), pp. 480–508, referred to as O.

[3] W. pp. 132–135 or O. pp. 483–486.

[4] W. p. 135 or O. p. 488.

[5] Ore (O. p. 488) uses the term *similar*. We prefer *equivalent* since *similar* has a different meaning in connection with the theory of algebras. See p. 204.

The preceding properties hold for a very wide class of domains of integrity, namely, the proper Euclidean domains as defined by Wedderburn.[6] We shall now discuss certain properties peculiar to $\mathfrak{F}'[x]$.

Let $\mathfrak{C}[x]$ denote the *centrum* of $\mathfrak{F}'[x]$ i.e. the set of elements of $\mathfrak{F}'[x]$ commutative with every element in $\mathfrak{F}'[x]$. If a polynomial Π is commutative with x, its coefficients $\subset \mathfrak{F}$ and if it commutes with a, it contains only terms in the powers of x^r. It follows that $\mathfrak{C}[x]$ consists of all polynomials in x^r with coefficients $\subset \mathfrak{F}$.

A domain of integrity has been called *Hamiltonian* by Wedderburn[7] if for every element P there exists a $\bar{P}$, an *adjoint* of P such that $P\bar{P} = \Pi \subset$ centrum of the domain. It follows that $\bar{P}P = \Pi$.

THEOREM 2. *$\mathfrak{F}'[x]$ is a Hamiltonian domain of integrity.*

For any P,

$$1P = P1, \qquad x^rP = Px^r, \cdots, x^{kr}P = Px^{kr} \cdots .$$

By division obtain

$$x^{kr} = PU_k + V_k. \tag{4}$$

Since the polynomials $V_0, V_1, V_2, \cdots$ all have $\delta < \delta(P)$, they can not be linearly independent with respect to $\mathfrak{F}$. There exists $\alpha_0, \alpha_1, \cdots \alpha_f$ not all zero such that

$$\sum_{k=0}^{f} \alpha_k V_k = 0. \tag{5}$$

From (4) and (5) we have

$$\Pi = \Sigma\alpha_k x^{kr} = P(\Sigma\alpha_k U_k) = P\bar{P}. \tag{6}$$

Since there exists a division process in $\mathfrak{C}[x]$, it follows easily that the polynomial Π of lowest degree (and hence the adjoint of lowest degree) is unique except for a multiplier $\subset \mathfrak{F}$. If the inverses of the polynomials $\mathfrak{C}[x]$ are adjoined to $\mathfrak{F}'[x]$, we obtain a division algebra containing $\mathfrak{F}'[x]$.[8]

THEOREM 3. *Let $\Pi \subset \mathfrak{C}[x]$ and be irreducible in $\mathfrak{C}[x]$. If $\Pi = P_1P_2 \cdots P_s$ where the P's are irreducible in $\mathfrak{F}'[x]$, then all the P's are equivalent.*

Suppose Π has the form (6) and $\alpha_f = 1$, $\alpha_0 \neq 0$.[9] Let Q be an irreducible right factor of Π with leading coefficient 1, i.e.

$$\Pi = RQ. \tag{7}$$

We may transform (7) by an element of $\mathfrak{F}'$ or by x obtaining

$$\Pi = b\Pi b^{-1} = (bRb^{-1})(bQb^{-1}) = x\Pi x^{-1} = (xRx^{-1})(xQx^{-1}), \quad b \subset \mathfrak{F}'. \tag{8}$$

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The transform xRx^{-1} is obtained by taking conjugates of the coefficients of R. We indicate generally the result of any transformation as (8) by $\Pi = R'Q'$. It follows that $Q^* = [Q', Q]_L Q^{-1}$ is a right factor of R. Q^* is a unit if and only if Q' and Q differ by unit factors, which means in this case that $Q' = Q$. But, if $Q' = Q$ for every transformation, then $Q \subset \mathfrak{C}[x]$ and from the assumption of the irreducibility of Π in $\mathfrak{C}[x]$ it follows that $\Pi = Q$. In the contrary case we obtain a new factor Q^*, equivalent to Q' and hence to Q such that

$$\Pi = R_1 Q^* Q = R_1 Q_1.$$

We suppose Q_1 normalized so that its leading coefficient is 1 and then we use the same process on it as used before on Q. We obtain $Q_1^* = [Q_1, Q_1']_L Q^{-1}$ which has factors equivalent to Q unless $Q_1 = Q_1'$.[10] Continuing in this way we finally reach a stage at which $Q_k = Q_k'$ for every transform. As before, it follows that $\Pi = Q_k$. Thus, we obtain a factorization of the type desired. It follows then from Theorem 1 that any factorization has the desired properties.

2. Residue Classes in $\mathfrak{F}'[x]$

In this section we consider the properties of $\mathfrak{F}'[x]$ as an algebra over $\mathfrak{F}$.

The sub-algebra $\mathfrak{F}'[x]P$ (the set of elements of the form KP, $K \subset \mathfrak{F}'[x]$) is left-invariant in $\mathfrak{F}'[x]$. One may divide the elements of $\mathfrak{F}'[x]$ into classes modulo $\mathfrak{F}'[x]P$ or (mod P). The set of these classes forms a vector space Σ_P whose order is t^r over $\mathfrak{F}$ where $t = \delta(P)$. In order that Σ_P be an algebra it must be closed under multiplication, i.e. from

$$A \equiv A_1 \pmod{P}, \qquad B \equiv B_1 \pmod{P} \tag{9}$$

must follow

$$BA \equiv B_1 A_1 \pmod{P}. \tag{10}$$

But (9) and (10) imply

$$PA \equiv 0 \pmod{P} \quad \text{or} \quad PA = A'P. \tag{11}$$

Since A is arbitrary, it follows that a necessary and sufficient condition that Σ_P be an algebra is that (11) hold for every $A \subset \mathfrak{F}'[x]$. However, if we restrict ourselves to the sub-set, $\mathfrak{F}'[P]$, of elements A which satisfy (11), we may associate with every P a unique difference or residue algebra $\mathfrak{F}'(P) = \mathfrak{F}'[P] - \mathfrak{F}'[x]P$. $\mathfrak{F}'[P]$ is a domain of integrity and an algebra over $\mathfrak{F}$. It is in fact the maximal sub-algebra of $\mathfrak{F}'[x]$ which contains $\mathfrak{F}'[x]P$ as an invariant sub-algebra. We shall call $\mathfrak{F}'[P]$ the *normalizer*[11] of P and $\mathfrak{F}'(P)$ the *residue algebra* of P.

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[10] See O. p. 488–491.

[11] Ore, in Formale Theorie der linearen Differentialgleichungen II, Jour. fur Math., **168**, (1932) p. 242, uses the term *"Eigenring."* We use *normalizer* because of the analogy with the theory of groups. We refer to this paper of Ore's as O'.

Suppose that $A \subset \mathfrak{F}'[P]$ and $(A, P)_R = 1$. There exist an A_1 and a P_1 such that

$$A_1A + P_1P = 1. \tag{12}$$

By division of PA_1 by P on the right we get $PA_1 = A_1'P + A_2$ where $\delta(A_2) < \delta(P)$. From (12) we get

$$PA_1A = P(1 - P_1P) = (1 - PP_1)P \tag{13}$$

and from (11)

$$PA_1A = A_1'PA + A_2A = A_1'A'P + A_2A \tag{14}$$

and from (13) and (14)

$$A_2A = (1 - P_1P - A_1'A')P.$$

Then $P^* = [P, A]_L A^{-1}$ is a right factor of A, which is impossible because of the degrees of P^* and A_2 unless $A_2 = 0$, or $PA_1 = A_1'P$. Since $\mathfrak{F}'(P)$ is an algebra it follows that there is a P_2 such that $AA_1 + P_2P = 1$. We have immediately

THEOREM 4. *If P is irreducible, $\mathfrak{F}'(P)$ is a division algebra.*[12]

In the same way we may treat the algebra $P\mathfrak{F}'[x]$ and we may define the left-normalizer $\mathfrak{F}_L'[P]$ as the set of elements A' which satisfy (11) and we may define the left-residue algebra as $\mathfrak{F}_L'(P) = \mathfrak{F}'[P] - P\mathfrak{F}'[x]$. If we let A and A' be corresponding elements, (11) gives an isomorphism between $\mathfrak{F}'[P]$ and $\mathfrak{F}_L'[P]$ and also between $\mathfrak{F}'(P)$ and $\mathfrak{F}_L'(P)$.

We shall require later the trivial remark that $\mathfrak{F}'(P) \cong \mathfrak{F}'(bPc)$ where $b, c \subset \mathfrak{F}'$. This is verified immediately.

3. The Structure of Cyclic Algebras

A *cyclic algebra* is defined to be an algebra $\mathfrak{F}'(\Pi)$ where $\Pi = x^r - \gamma$ $(\gamma \neq 0)$. $\mathfrak{F}'(\Pi)$ is a normal simple algebra. We shall now discuss the structure of $\mathfrak{F}'(\Pi)$ and its relation to the factorization of Π in $\mathfrak{F}'[x]$.

Suppose that $\Pi = P\bar{P}$ where $(P, \bar{P})_R = 1$. Then from (3) we have $[P, \bar{P}]_L = \Pi$. There exists a Q and an R' such that

$$Q\bar{P} + R'P = 1 \tag{15}$$

where we may take $\delta(Q) < \delta(P)$. From (11) we obtain

$$A\bar{P} = \bar{P}A'. \tag{16}$$

Since $Q \subset \mathfrak{F}'[P]$, we may set

$$Q\bar{P} = \bar{P}Q' = e \tag{17}$$

[12] This theorem has been given by Ore (O' p. 242). Our proof, however, is different from his.

and since the rôles of P and $\bar{P}$ may be interchanged,

$$R'P = PR = \bar{e}. \tag{18}$$

Now, (15), (17) and (18) give $(P, \bar{P})_L = 1$ and as before

$$[P, \bar{P}]_R = \Pi. \tag{19}$$

(15), (17) and (18) imply also

$$e^2 \equiv e \pmod{\Pi}, \qquad \bar{e}^2 \equiv \bar{e} \pmod{\Pi}, \qquad e\bar{e} \equiv \bar{e}e \equiv 0 \pmod{\Pi}. \tag{20}$$

If $A \subset \mathfrak{F}'[P]$ from (15) and (17) follows

$$A = \bar{P}(Q'A) + (R'A')P = \bar{P}Y + ZP \tag{21}$$

and conversely any polynomial of the form $\bar{P}Y + ZP \subset \mathfrak{F}'[P]$. Thus, $\mathfrak{F}'[P]$ is identical with the set of these polynomials. If $\delta(A) < \delta(P)$, we may take $\delta(Y) < \delta(P)$ in (21).

As in §2 let Σ_P denote the set of elements of $\mathfrak{F}'[x]$ of $\delta < \delta(P) = t$. We have seen that Σ_P is a vector space whose order is rt over $\mathfrak{F}$. Let $\Sigma_P^{(1)} = \Sigma_P \wedge \mathfrak{F}'[P]$. $\Sigma_P^{(1)}$ is a vector space whose elements may be used as representatives of the elements of the residue algebra $\mathfrak{F}'(P)$. Finally, let $\Sigma_P^{(2)}$ be the sub-space of elements W of Σ_P such that $\bar{P}W = W^*P$ or, what amounts to the same thing, such that

$$W\bar{P} = PW^*. \tag{22}$$

We shall require later

LEMMA 1. $\Sigma_P = \Sigma_P^{(1)} + \Sigma_P^{(2)}$, $\Sigma_P^{(1)} \wedge \Sigma_P^{(2)} = 0$.

$\Sigma_P^{(1)} \wedge \Sigma_P^{(2)} = 0$. For, if W satisfies (22) and (16), then $\bar{P}W' = PW^*$ which is impossible because of (19) unless $W' = W^* = W = 0$. Now, (21) with $\delta(Y) < t$ gives a linear transformation of Σ_P into $\Sigma_P^{(1)}$ determined by dividing $\bar{P}Y$ on the right by P and letting Y correspond to A. $\Sigma_P^{(2)}$ is precisely the sub-space which goes into 0 under this transformation. Hence, order of Σ_P = order of $\Sigma_P^{(1)}$ + order of $\Sigma_P^{(2)}$ and from the previous result, $\Sigma_P = \Sigma_P^{(1)} + \Sigma_P^{(2)}$.

We prove next the fundamental

THEOREM 5. *If* $\Pi = P\bar{P} = \bar{P}P$, $(P, \bar{P})_R = 1$, *then* $\mathfrak{F}'(P) \cong e\mathfrak{F}'(\Pi)e$ *where* e *is given by* (17).

We recall the Peirce decomposition of $\mathfrak{F}'(\Pi)$ relative to the idempotent elements e and $\bar{e}$:

$$\mathfrak{F}'(\Pi) = e\mathfrak{F}'(\Pi)e + e\mathfrak{F}'(\Pi)\bar{e} + \bar{e}\mathfrak{F}'(\Pi)e + \bar{e}\mathfrak{F}'(\Pi)\bar{e}$$

or

$$S \equiv eSe + eS\bar{e} + \bar{e}Se + \bar{e}S\bar{e} \pmod{\Pi}.$$

eSe is divisible on both sides by $\bar{P}$ and if

$$eSe \equiv S_{11} \pmod{\Pi} \tag{23}$$

where $\delta(S_{11}) < r$, S_{11} is divisible on both sides by $\bar{P}$ i.e.

$$S_{11} = T_{11}\bar{P} = \bar{P}T'_{11}. \tag{24}$$

and so $S_{11} \subset \mathfrak{F}'[P]$. Similarly, we obtain

$$S_{12} = T^*_{12}P = \bar{P}T_{12}, \qquad S_{21} = T_{21}\bar{P} = PT^*_{21}, \qquad S_{22} = T'_{22}P = PT_{22} \tag{25}$$

where $\delta(S_{ik}) < r$ and

$$S \equiv S_{11} + S_{12} + S_{21} + S_{22} \pmod{\Pi}.$$

Now, if $\delta(S) < r$, from considerations of degree follows

$$S = S_{11} + S_{12} + S_{21} + S_{22}. \tag{26}$$

Conversely, suppose S is any element of $\delta < r$ and let (26) be an expression for S where we assume merely that the S_{ik} satisfy (24) and (25) and $\delta(S_{ik}) < r$. Then, this expression is the Peirce decomposition of S. To show this we have merely to show that an expansion of this type is unique. Suppose then $S = 0$ where S has the form (26) the S_{ik} satisfying (24) and (25). Then

$$\bar{P}(T'_{11} + T_{12}) = -P(T^*_{21} + T_{22})$$

which according to (19) implies $S_{11} + S_{12} = 0 = S_{21} + S_{22}$ and by the same kind of argument applied to $S_{11} + S_{12}$, $S_{11} = S_{12} = S_{21} = S_{22} = 0$ or (26) is unique.

We shall show that the correspondence between the class of $eSe \pmod{\Pi}$ and the class of $S_{11} \pmod{P}$, where S_{11} and eSe satisfy (23), is an isomorphism between $e\mathfrak{F}'(\Pi)e$ and $\mathfrak{F}'(P)$.

In the first place, the elements S_{11} obtained by (23) are a complete set of representatives of the elements of $\mathfrak{F}'(P)$. For, let T_{11} be any element of $\Sigma^{(1)}_P$. Then $S_{11} = T_{11}\bar{P} = \bar{P}T'_{11}$ has $\delta < r$ and according to the Peirce decomposition, (26), $S_{11} \subset e\mathfrak{F}'(\Pi)e$. As T_{11} takes on all values in $\Sigma^{(1)}_P$, S_{11} generates a complete set of residues (mod P) since $\overline{P}$ has an inverse, Q, (mod P). Hence, the correspondence between classes under consideration is $(1-1)$. It is evidently preserved under addition and scalar multiplication. Let

$$S^{(1)}_{11}S^{(2)}_{11} \equiv S^{(3)}_{11} \pmod{P}.$$

From (17) we have

$$eS^{(1)}_{11}S^{(2)}_{11}e \equiv eS^{(3)}_{11}e \pmod{\Pi} \tag{27}$$

and by (15)

$$eS^{(1)}_{11}S^{(2)}_{11}e = eS^{(1)}_{11}(e + \bar{e})S^{(2)}_{11}e = eS^{(1)}_{11}eS^{(2)}_{11}e + eS^{(1)}_{11}\bar{e}S^{(2)}_{11}e. \tag{28}$$

Further,

$$eS^{(1)}_{11}\bar{e}S^{(2)}_{11}e = \bar{P}Q'S^{(1)}_{11}R'PS^{(2)}_{11}\bar{P}Q' = \bar{P}Q'S^{(1)}_{11}R'S^{(2)}_{11}\Pi Q' \equiv 0 \pmod{\Pi} \tag{29}$$

and so from (27), (28), (29) and (20) follows

$$eS_{11}^{(3)}e \equiv (eS_{11}^{(1)}e)\,(eS_{11}^{(2)}e) \pmod{\Pi}$$

i.e. the correspondence is an isomorphism.

If we allow S_{12} in (25) to correspond to T_{12}, we obtain a non-singular linear transformation of the elements of $e\mathfrak{F}'(\Pi)\bar{e}$ into the elements of $\Sigma_P^{(2)}$. For, if $T_{12} \subset \Sigma_P^{(2)}$, as before, $S_{12} = \bar{P}T_{12} \subset e\mathfrak{F}'(\Pi)\bar{e}$ so that the correspondence is defined over the whole of $\Sigma_P^{(2)}$. It is evidently (1–1) and linear with respect to $\mathfrak{F}$. Hence, we have

LEMMA 2. *The orders of $e\mathfrak{F}'(\Pi)\bar{e}$ and $\Sigma_P^{(2)}$ over $\mathfrak{F}$ are equal.*

It follows from Theorem 5 that $\mathfrak{F}'(P)$ is a normal simple algebra *similar to* ($\sim$) $\mathfrak{F}'(\Pi)$, i.e. their division algebras in the Wedderburn decomposition are isomorphic. If P is irreducible, $\mathfrak{F}'(P)$ is the division algebra part of $\mathfrak{F}'(\Pi)$ (in the sense of isomorphism) and e is a primitive idempotent element.

Theorem 5 holds also in the more general case in which $\Pi \subset \mathfrak{C}[x]$ and is irreducible there. However, it can be shown without difficulty that $\mathfrak{F}'(\Pi)$ is cyclic over its centrum and so from the point of view of the theory of algebras there is no gain in generality in this case. We return to our previous consideration of $\mathfrak{F}'(\Pi)$, $\Pi = x^r - \gamma$.

LEMMA 3. *If $\Pi = P\bar{P}$, P irreducible, then there exists an associate, P_1, of P such that $\Pi = P_1\bar{P}_1$ and $(P_1, \bar{P}_1)_R = 1$.*

If $(P, \bar{P})_R \neq 1$, then $\bar{P} = TP = PT$ and

$$\bar{P}^2 \equiv 0 \pmod{\Pi}. \tag{30}$$

Let b and c be arbitrary elements of $\mathfrak{F}'$. Then

$$\Pi = (bPc)\,(c^{-1}Pb^{-1}) = \bar{P}_1P_1 = P_1\bar{P}_1.$$

If the lemma is false, then (30) holds for every $\bar{P}_1$. A direct calculation shows that $\bar{P}_1$ is divisible on both sides by P and so

$$\bar{P}_1\bar{P} \equiv \bar{P}\bar{P}_1 \equiv 0 \pmod{\Pi} \tag{31}$$

and in fact (31) holds for any pair of associates of P. If x^k is a power of x actually present in $\bar{P}$, $a^{(k)}\bar{P} - \bar{P}a$ has one term less than P and by (30) and (31) is nilpotent. Continuing in this way we may obtain a linear combination of terms as $\bar{P}_1 = x^l (l < r)$ which is nilpotent. But this is impossible since $\gamma \neq 0$.

It follows from Theorem 5, Lemma 3 and the remark at the end of §2 that if $\Pi = P\bar{P}$, P irreducible, $\mathfrak{F}'(P) \cong$ division algebra part of $\mathfrak{F}'(\Pi)$.

THEOREM 6. *If P is an irreducible factor of Π of degree t, the order of the division algebra part of $\mathfrak{F}'(\Pi)$ is t^2.*

It is only necessary to compute the order of $\mathfrak{F}'(P)$ where P is irreducible and $(P, \bar{P})_R = 1$. Let n^2 denote this order. By Lemma 1 $n^2 = rt - h$ where $h =$ order of $\Sigma_P^{(2)}$. The order of the matric part (e_{ik}) of $\mathfrak{F}'(\Pi)$ is then

$$m^2 = \frac{r^2}{n^2}. \tag{32}$$

h is also the order of $e\mathfrak{F}'(\Pi)\bar{e}$ according to Lemma 2. If e is taken as e_{11} in the matric part of $\mathfrak{F}'(\Pi)$, then $e\mathfrak{F}'(\Pi)\bar{e}$ is the direct product of the division algebra part $\mathfrak{F}'(\Pi)$ and the algebra whose basis is $(e_{12}, e_{13}, \cdots, e_{1m})$ over $\mathfrak{F}$. It follows that

$$m^2 = \left(\frac{h}{rt-h}+1\right)^2 = \left(\frac{rt}{rt-h}\right)^2 = \frac{r^2t^2}{n^4}. \tag{33}$$

From (32) and (33) we find $n = t$.

Theorem 6 gives an indirect proof that the degrees of the irreducible factors of Π are equal. This is also a consequence of Theorem 3.

We summarize the results of this section in the following

THEOREM 7. *If $\Pi = P_1P_2\cdots P_s$ where the P's are irreducible, all the P's have the same degree. If this common degree is t, $\mathfrak{F}'(\Pi)$ has division algebra of order t^2 and matric algebra of order s^2. Further, $\mathfrak{F}'(\Pi) \sim \mathfrak{F}'(P_i)$.*

4. Norm Conditions

We shall now derive a criterion based on Theorem 7 in order that the division algebra part of $\mathfrak{F}'(\Pi)$ have order t^2. From this criterion the norm conditions for cyclic algebras follow as immediate corollaries.

If $P \subset \mathfrak{F}'[x]$, evidently

$$P = p_{11} + p_{12}x + \cdots + p_{1r}x^{r-1}$$

where the p_{1i} are polynomials in x^r with coefficients $\subset \mathfrak{F}'$. This expression for P is unique. In the same way let

$$x^{k-1}P = p_{k1} + p_{k2}x + \cdots + p_{kr}x^{r-1} \qquad (k = 1, \cdots r). \tag{34}$$

If $\mathfrak{X}$ denotes the matrix

$$\begin{pmatrix} 1 \\ x \\ \vdots \\ x^{r-1} \end{pmatrix}$$

and $\mathfrak{P}$ the matrix (p_{ij}), then the equations (34) may be written as

$$\mathfrak{X}P = \mathfrak{P}\mathfrak{X}. \tag{35}$$

The correspondence of P into $\mathfrak{P}$ is readily verified to be an isomorphism.

THEOREM 8. *The characteristic equation of $\mathfrak{P}$ has coefficients which are polynomials in $\mathfrak{C}[x]$.*

If we allow the p_{1i} in the expression for P to have denominators $\subset \mathfrak{C}[x]$, then P represents a general element of the quotient algebra of $\mathfrak{F}'[x]$. This algebra has as its centrum the field $\mathfrak{K} = \mathfrak{F}(x^r = z)$ and is in fact the cyclic algebra $\mathfrak{K}'(\Omega)$ where $\mathfrak{K}' = \mathfrak{K}(a)$ and $\Omega = x^r - z$. The matrices $\mathfrak{P}$ give a representation of $\mathfrak{K}'(\Omega)$. The characteristic equation of $\mathfrak{P}$ is the minimal equation of the algebra $\mathfrak{K}'(\Omega)$. Hence, the coefficients of this equation $\subset \mathfrak{K}$.

Obviously, if $P \subset \mathfrak{F}'[x]$, the characteristic equation of $\mathfrak{P}$ has integral coefficients in $\mathfrak{K}$, i.e. $\subset \mathfrak{C}[x]$.

In particular, the determinant of $\mathfrak{P} \subset \mathfrak{C}[x]$. We shall call this determinant the *norm* of P, $N(P)$. If $b \subset \mathfrak{F}'$, $N(b)$ is the ordinary norm, the product of the conjugates of b. For $\Pi \subset \mathfrak{C}[x]$, $N(\Pi) = \Pi^r$. Finally, if

$$P = a_0 + a_1 x + \cdots + a_t x^t, \qquad a_i \subset \mathfrak{F}', \qquad t < r,$$

then

$$\mathfrak{P} = \begin{bmatrix} a_0 & a_1 & \cdots & a_t & \cdots & 0 \\ 0 & a_0^{(1)} & & & a_t^{(1)} & \vdots \\ \vdots & \vdots & & & & \vdots \\ 0 & \vdots & & & & a_t^{(r-t-1)} \\ a_t^{(r-t)}x^r & 0 & & a_0^{(r-t)} & & a_{t-1}^{(r-t)} \\ & \vdots & & & & \vdots \\ & & a_t^{(r-1)}x^r & \cdots & \cdots & a_0^{(r-1)} \end{bmatrix}$$

and so

$$N(P) = (-1)^t N(-1)^t N(a_t) x^{tr} + \cdots + N(a_0). \tag{36}$$

For any P we have

$$(\text{adjoint } \mathfrak{P})\mathfrak{P} = N(P)\mathfrak{J}$$

where $\mathfrak{J}$ is the unit matrix and hence from (35)

$$(\text{adjoint } \mathfrak{P})\mathfrak{X}P = N(P)\mathfrak{X}. \tag{37}$$

By comparing elements in the first row of the matrices on the two sides of the equality sign in (37), we find that P is a factor of $N(P)$. Further

$$N(PQ) = N(P)N(Q). \tag{38}$$

We may now prove the following

THEOREM 9. *If t is the smallest positive integer such that $\alpha\Pi^t = N(P)$ for some $\alpha \subset \mathfrak{F}$ and $P \subset \mathfrak{F}'[x]$, then the division algebra part of $\mathfrak{F}'(\Pi)$ has order t^2 and is isomorphic to $\mathfrak{F}'(P)$.*

Let P, α and t satisfy the conditions of the theorem. Then P is irreducible. For, suppose $P = P_1P_2$ where $\delta(P_1) = t_1 < t$. From (38) follows $\alpha\Pi^t = N(P_1)N(P_2)$. $N(P_1)$ and $N(P_2) \subset \mathfrak{C}[x]$ and have P_1 and P_2 respectively as factors. Since Π is irreducible in $\mathfrak{C}[x]$, we must have $N(P_1) = \beta\Pi^u$ where $u < t$ and $\beta \subset \mathfrak{F}$. This contradicts our assumption about t.

P is a right factor of $\alpha\Pi^t$ and if not a factor of Π, $(P, \Pi)_R = 1$. Hence, $P =$

$[P, \Pi]_L \Pi^{-1}$ is a factor of $\alpha \Pi^{t-1}$. By repeating this argument sufficiently often, we are lead to a contradiction.

Thus, P is an irreducible factor of Π and hence $\delta(P) \leqq r$. From (36) we have $\delta(P) = t$ and the theorem follows from Theorem 7.

If the leading coefficient of P is a_t, then $(-1)^t a_t^{-1} P$ also satisfies the condition of the theorem and has $(-\Pi)^t$ for its norm.

THEOREM 10. *A necessary and sufficient condition that* $\mathfrak{F}'(\Pi)$ *be matric is that* γ *be the norm of an element of* $\mathfrak{F}'$.[13]

By Theorem 9 the condition is that

$$\gamma - x^r = N(m - x) = N(m) - x^r.$$

Hence, $\gamma = N(m)$. By retracing the steps we get the converse.

THEOREM 11. *If* q *is the smallest positive integer for which* γ^q *is the norm of an element of* $\mathfrak{F}'$, *then the division algebra part of* $\mathfrak{F}'(\Pi)$ *has order* $\geqq q^2$.

If the order of the division algebra is t^2, then according to Theorem 9, there exists a P such that $N(P) = (\gamma - x^r)^t$ where we assume that the leading coefficient of P is $(-1)^t$ and last coefficient is a_0. From (36) follows

$$(-1)^t x^{tr} + \cdots + N(a_0) = (\gamma - x^r)^t$$

and hence $\gamma^t = N(a_0)$ so that $t \geqq q$.[14]

If $r = st$, there is a unique sub-field $\mathfrak{F}_t$ of $\mathfrak{F}'$ of order t over $\mathfrak{F}$. $\mathfrak{F}_t$ is in fact the sub-field of $\mathfrak{F}'$ which is invariant under the automorphism which sends a into $a^{(t)}$. If $b \subset \mathfrak{F}_t$, we denote $bb^{(1)} \cdots b^{(t-1)}$ by $N_t(b)$. By means of $\mathfrak{F}_t$ and the automorphism which sends any b of $\mathfrak{F}_t$ into $b^{(1)}$ we can define a cyclic polynomial domain $\mathfrak{F}_t[x]$. If P is a polynomial in $\mathfrak{F}_t[x]$, we denote its residue algebra in $\mathfrak{F}_t[x]$ by $\mathfrak{F}_t(P)$. We require the following

LEMMA. *If* $P = x^t - \delta$, $\delta \subset \mathfrak{F}$, *then* $\mathfrak{F}'(P) \cong \mathfrak{F}_t(P)$ *and is a cyclic algebra.*

For, if $A = a_0 + a_1 x + \cdots + a_{t-1} x^{t-1}$ and $PA = A'P$, then the $a_i \subset \mathfrak{F}_t$.

We may now prove a result of Albert's:[15]

THEOREM 12. *If* $r = st$ *and* $\gamma^s = N(c)$, $c \subset \mathfrak{F}'$, *then there exists an element* $d \subset \mathfrak{F}_t$ *such that* $\gamma = N_t(d)$.

By the hypothesis and by Theorem 10 we conclude that the cyclic algebra $\mathfrak{F}'(x^r - \gamma^s)$ is a matric algebra. But

$$x^r - \gamma^s = (x^t - \gamma)(x^{t(s-1)} + \gamma x^{t(s-2)} + \cdots + \gamma^{s-1}) = P\bar{P}$$

where $P = x^t - \gamma$. Evidently $(P, \bar{P}) = 1$ and hence by Theorem 5, $\mathfrak{F}'(x^r - \gamma^s) \sim \mathfrak{F}'(x^t - \gamma) \cong \mathfrak{F}_t(x^t - \gamma)$. Thus $\mathfrak{F}_t(x^t - \gamma)$ is matric and so by Theorem 10

$$\gamma = N_t(d) \qquad d \subset \mathfrak{F}_t.$$

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[13] cf. Helmut Hasse, Theory of cyclic algebras over an algebraic number field, Trans. Am. Math. Soc., **34**, (1932), p. 199, or A. A. Albert, On the construction of cyclic algebras with a given exponent, Am. Jour. of Math., **54**, (1932), p. 10. These papers will be denoted by H. and by A. respectively.

[14] cf. H. p. 194 and A. p. 10.

[15] A. p. 8.

The following theorem also has been given by Albert:[16]

THEOREM 13. *A cyclic algebra is a direct product of cyclic algebras of prime power orders.*

We recall the well known theorem that the direct product $\mathfrak{F}'(x^r - \gamma_1) \times \mathfrak{F}'(x^r - \gamma_2) \sim \mathfrak{F}'(x^r - \gamma_1\gamma_2)$.[17] Now, suppose that $r = st$, $(s, t) = 1$ and determine l, m and k so that

$$ls + mt = 1 + kr \qquad l, m, k > 0.$$

Then

$$\mathfrak{F}'(x^r - \gamma^{ls}) \times \mathfrak{F}'(x^r - \gamma^{mt}) \sim \mathfrak{F}'(x^r - \gamma) \times \mathfrak{F}'(x^r - \gamma^{kr}).$$

But by Theorem 10, $\mathfrak{F}'(x^r - \gamma^{kr})$ is matric and hence

$$\mathfrak{F}'(x^r - \gamma) \sim \mathfrak{F}'(x^r - \gamma^{ls}) \times \mathfrak{F}'(x^r - \gamma^{mt}).$$

On the other hand, since

$$x^r - \gamma^{ls} = (x^t - \gamma^l)\,(x^{t(s-1)} + \gamma^l x^{t(s-2)} + \cdots + \gamma^{l(s-1)}) = P\bar{P}$$

where $P = x^t - \gamma^l$ and $(P, \bar{P}) = 1$, it follows as in the proof of Theorem 12 that $\mathfrak{F}'(x^r - \gamma^{ls}) \sim \mathfrak{F}_t(x^t - \gamma^l)$ and similarly $\mathfrak{F}'(x^r - \gamma^{mt}) \sim \mathfrak{F}_s(x^s - \gamma^m)$. Hence

$$\mathfrak{F}'(x^r - \gamma) \sim \mathfrak{F}_t(x^t - \gamma^l) \times \mathfrak{F}_s(x^s - \gamma^m).$$

By comparing the degrees of the algebras involved we have

$$\mathfrak{F}'(x^r - \gamma) \cong \mathfrak{F}_t(x^t - \gamma^l) \times \mathfrak{F}_s(x^s - \gamma^m).$$

If s or t is not a power of a prime we repeat the argument using $\mathfrak{F}_s(x^s - \gamma^m)$ or $\mathfrak{F}_t(x^t - \gamma^l)$ in place of $\mathfrak{F}'(x^r - \gamma)$. Continuing in this way we obtain the theorem.

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[16] A. p. 9.

[17] A. p. 6.

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A NOTE ON NON-COMMUTATIVE POLYNOMIALS

BY NATHAN JACOBSON

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Ö. Ore[1] has recently considered polynomials in an indeterminate x whose coefficients (taken on the left) lie in a non-commutative field or division algebra $\mathfrak{A}$ where

$$xa = \bar{a}x + a' \qquad a, \bar{a}, a' \subset \mathfrak{A} \tag{1}$$

He has shown that if the associative and distributive postulates are assumed, the set of these polynomials constitutes a domain of integrity, $\mathfrak{A}[x]$, and the operations $(^-)$ and $(')$ in $\mathfrak{A}$ satisfy the conditions:

$$\overline{(a+b)} = \bar{a} + \bar{b} \qquad \overline{(ab)} = \bar{a}\bar{b}$$

$$(a+b)' = a' + b' \qquad (ab)' = \bar{a}b' + a'b.$$

For every element $A \neq 0$, $\subset \mathfrak{A}[x]$ there is defined a real non-negative number $\delta(A)$, (the degree of A) such that

(I) $\qquad \delta(AB) = \delta(A) + \delta(B) \qquad \delta(A+B) \leqq \text{Max. } (\delta(A), \delta(B))$

(II) No infinite sequence $A_1, A_2, A_3 \cdots$ can exist such that

$$\delta(A_1) > \delta(A_2) > \delta(A_3) > \cdots$$

(III) For every pair A, B there exists a Q and an R such that $A = QB + R$ where either $\delta(R) < \delta(B)$ or $R = 0$.

A domain of integrity satisfying (II) and (III) will be called *right-Euclidean*.[2] The property (I) will be referred to as the *non-Archimedian* property of δ.[3]

It is the purpose of the present note to prove the converse of these results and thus obtain an abstract characterization of polynomial domains.

THEOREM. *A domain of integrity in which there is defined a non-negative function δ having the properties (I), (II) and (III) is a polynomial domain.*

The condition $\delta(a) = 0$ is necessary and sufficient in order that a be a unit.[4]

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[1] Ö. Ore, Theory of non-commutative polynomials, Annals of Math., **34** (1933) pp. 480–508.

[2] J. H. M. Wedderburn, in Non-commutative domains of integrity, Jour. für Math., **167** (1932) p. 132, calls a domain Euclidean if two sided division exists and (II) is satisfied in the domain.

[3] A. Ostrowski, Über einige Lösungen der Funktionalgleichung $\varphi(x)\varphi(y) = \varphi(xy)$, Acta Math., **41** (1916–1917), p. 272.

[4] cf. Wedderburn, loc. cit.

For it follows from (I) that $\delta(1) = 0$ and since $\delta(aa^{-1}) = \delta(a) + \delta(a^{-1}) = 0$ then $\delta(a) = 0 = \delta(a^{-1})$. Conversely, if $\delta(a) = 0$, (III) gives $1 = qa + r$ and since $\delta(a) = 0$, we must have $r = 0$ so that $q = a^{-1}$. (I) shows that the set of units is closed under rational operations and thus forms a non-commutative field $\mathfrak{A}$.

Let x be an element of the domain for which $\delta(x)$ has the minimum positive value. The existence of such an element follows from (II). Now, there is no element P for which

$$n\delta(x) < \delta(P) < (n + 1)\delta(x) \tag{2}$$

where n is a positive integer. For, if we obtain by division

$$P = a_0x^n + P_1 \tag{3}$$

where $\delta(P_1) < \delta(x^n)$, then it follows from (I) that $\delta(P) \leqq \delta(a_0x^n)$. But if P_1 is transposed to the side of P in (3) and (I) is again applied, we obtain $\delta(a_0x^n) \leqq \delta(P)$ and hence

$$\delta(P) = \delta(a_0x^n) = \delta(a_0) + n\delta(x) \tag{4}$$

and from (2) follows $0 < \delta(a_0) < \delta(x)$ contrary to the assumption on x.

Now, if $\delta(P) = n\delta(x)$, (4) shows that $\delta(a_0) = 0$ and so $a_0 \subset \mathfrak{A}$. By repeating the argument for P_1 and using induction as allowed by (II), we obtain finally.

$$P = a_0x^n + a_1x^{n-1} + \cdots + a_n \tag{5}$$

where the $a_i \subset \mathfrak{A}$. It is evident from considerations of degree that the expression for P given by (5) is unique, i.e. x is transcendental over $\mathfrak{A}$.

The conditions (I) imply (1) and the proof is complete.

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ON CERTAIN IMPRIMITIVE FIELDS OF DEGREE p^2 OVER P OF CHARACTERISTIC p*

BY A. ADRIAN ALBERT

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1. **Introduction.** E. Artin and O. Schreier have determined all cyclic fields of degree p, p^2 over a field P of characteristic p and have obtained very beautiful results.[1] Their latter problem on fields of degree p^2 is but a part of the more general problem of determining all imprimitive fields

$$P(x) > P(u) > P$$

where $P(x)$ is cyclic of degree p over $P(u)$ which is itself cyclic of degree p over P.

In the present paper I apply the Artin-Schreier results to solve this more general problem. In case $p = 2$ more explicit results are obtainable[2] and this application is also made here.

2. **The Artin-Schreier theory.** Let P be any infinite field of characteristic p. Then we make the

DEFINITION. *A quantity α of P is called a p-difference of P if*

$$\alpha = \lambda^p - \lambda \tag{1}$$

for λ in P.

It is well known that cyclic equations of degree p over P are not solvable by radicals. Thus the usual criterion that a quantity α is not the pth power of a quantity of P is of no value for questions of irreducibility in P. But E. Artin and O. Schreier have proved

THEOREM 1. *A normed equation*

$$\phi(\omega) \equiv \omega^p - \omega - \alpha = 0 \qquad (\alpha \text{ in } P), \tag{2}$$

is cyclic if and only if α is not a p-difference of P. Every cyclic field C of degree p over P has a generator u which satisfies (2), $C = P(u)$, and in fact

$$\phi(\omega) \equiv (\omega - u)\,[\omega - (u + 1)] \cdots [\omega - (u + p - 1)]. \tag{3}$$

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[1] Hamburg Abhandlungen, vol. 5 (1926–27), pp. 225–231.

[2] These results are of great importance in the problem of determining all normal division algebras of degree 4 over F of characteristic 2. I have solved this problem in a paper which is being offered to the American Journal of Mathematics for publication.

Artin-Schreier have also proved

THEOREM 2. *A primitive quantity v of $C = P(u)$ satisfies a normed equation if and only if*

$$v = ku + \gamma \qquad (k = 1, 2, \cdots, p-1), \tag{4}$$

where γ is in P.

THEOREM 3. *Let $C = P(u)$ be cyclic of degree p over P, u a root of (2). Then there exist cyclic fields of degree p^2 over P containing C. Every such field is generated by a root x of*

$$\psi(\omega) \equiv \omega^p - \omega - f = 0, \tag{5}$$

where $f = f(u) = g(u) + \lambda$, λ is any quantity of P, $g(\eta)$ is the uniquely determined polynomial

$$g(\eta) \equiv \gamma_1\eta^{p-1} + \gamma_2\eta^{p-2} + \cdots + \gamma_{p-1}\eta \tag{6}$$

satisfying identically

$$g(\eta + 1) - g(\eta) \equiv (\eta + \alpha)^{p-1} - \eta^{p-1}. \tag{7}$$

Conversely if x satisfies (5) with (7) holding then $P(x) > C = P(u)$ is cyclic of degree p^2 over P.

In fact Artin-Schreier showed that the cyclic equation of degree p^2 over P satisfied by x is

$$\begin{aligned}\Phi(\omega) &\equiv [\omega^p - \omega - f(u)][\omega^p - \omega - f(u+1)] \cdots [\omega^p - \omega - f(u+p-1)] \\ &\equiv (\omega - x)(\omega - x^s) \cdots (\omega - x^{s^{m-1}}),\ m = p^2,\ x^{s^m} = x,\end{aligned} \tag{8}$$

and

$$x^{s^\nu} \equiv x + u^{p-1} + (u+1)^{p-1} + \cdots + (u+\nu-1)^{p-1} \qquad (\nu = 1, 2, \cdots), \tag{9}$$

while in particular

$$x^{s^p} = x + u^{p-1} + \cdots + (u+p-1)^{p-1} = x - 1. \tag{10}$$

Artin-Schreier did not consider the question of the equivalence of the fields defined by Theorem 3. We therefore add to their theory the following simple discussion.

Let $P(x)$ amd $P(y)$ be cyclic of degree p^2 over P. If $P(x) = P(y)$ then they must evidently have the same cyclic $C = P(u)$ of degree p in common. Hence, by Theorem 3,

$$x^p - x = a, \qquad y^p - y = b, \tag{11}$$

where, since $a = g + \lambda_1$, $b = g + \lambda_2$, and g satisfies (7), we have

$$b = a + \mu, \qquad \mu \text{ in } P. \tag{12}$$

The field $P(x)$ is cyclic of degree p over C. If y is in $P(x)$ then, by Theorem 2,

$$y = kx + c, \qquad c \text{ in } C. \tag{13}$$

But then $y^p - y = k^p x^p + c^p - kx - c = k(x + a) - kx + c^p - c$ since $k^p = k$ for any integer k of P. Then

$$c^p - c = ta + \mu \qquad (t = 1 - k), \tag{14}$$

since $y^p - y = a + \mu$.

Suppose first that $t \neq 0$. Then $t^{p-1} \equiv 1 (\mathrm{mod}\ p)$, so that, in P, $t^{p-1} = 1$. Also $(t^{p-2})^p = t^{p-2}$ and hence if $\mu_0 = t^{p-2}\mu$, $d = t^{p-2}C$, we have

$$d^p - d = t^{p-2}(c^p - c) = a + \mu_0. \tag{15}$$

The quantity $a + \mu_0$ differs from g by a quantity of P. By Theorem 3 if d is a root of (15) the field $P(d)$ is cyclic of degree p^2 over P contrary to our hypothesis that c, and hence $d = t^{p-2}c$, are in $P(u)$.

Hence $t = 0$, $k = 1$, $y = x + c$, $c^p - c = \mu$ in P. But c is in $P(u)$. By Theorem 2

$$c = hu + \gamma \qquad (\gamma \text{ in } P),$$

where h is an integer. Then

$$c^p - c = h^p u^p + \gamma^p - (hu + \gamma) = h\alpha + \gamma^p - \gamma = \mu,$$

and hence

$$\mu - h\alpha = \gamma^p - \gamma \tag{16}$$

is a p-difference of P.

Conversely let $a = b + \mu$ and let an integer h exist such that $\mu - h\alpha$ is a p-difference of P. Then $y_0 = x + hu + \gamma$ satisfies

$$y_0^q - y_0 = x^p + hu^p + \gamma^p - x - hu - \gamma = a + h\alpha + \mu - h\alpha = a + \mu = b$$

if h is an existing solution of (16). But then $P(y)$ and $P(x) = P(y_0)$ are equivalent. We have proved

THEOREM 4. *Let $P(x)$ and $P(y)$ be cyclic of degree p^2 over P with a common sub-field $P(u)$ of degree p over P so that we may assume that*

$$u^p = u + \alpha, \qquad x^p = x + f, \qquad y^p = y + f + \mu, \tag{17}$$

where f is as in Theorem 3 and μ is in P. Then $P(x)$ and $P(y)$ are equivalent if and only if there exists an integer h such that $\mu - h\alpha$ is a p-difference of P.

Suppose now that y satisfies a normed equation

$$\omega^p - \omega - b = 0, \tag{18}$$

where b is in $C = P(u)$. Assume also that $P(y)$ is cyclic of degree p^2 over P. Then $P(y) = P(x)$ where $x^p = x + f$ and f satisfies the conditions of Theorem 3. By Theorem 2, $y = kx + c$, c in C,

$$b = y^p - y = k^p x + c^p - kx - c = k(x + f) - kx + c^p - c,$$

so that $b - kf = c^p - c$. If $k = 0$ then (18) is reducible contrary to our hypothesis that $P(y)$ has degree p^2. Hence $k \neq 0$, $h = k^{p-2}$ is an integer with the property that $hb = k^{p-1}f + (hc)^p - (hc) = f + d^p - d$ and we have proved the existence of an integer $h \neq 0$ such that hb differs from f by a p-difference of C.

Conversely if $hb = f + c^p - c$ then $x = hy - c$ satisfies

$$\begin{aligned} x^p - x = h^p y^p - c^p - hy + c &= h(y + b) - hy + c - c^p \\ &= f + c^p - c + c - c^p = f, \end{aligned}$$

so that $P(x) = P(y)$ is cyclic of degree p^2 over P. Since $P(y)$ has degree at most p^2 over P we have $P(y) = P(x)$ and have proved

THEOREM 5. *Let $C = P(u)$ be cyclic of degree p over P, g be defined as in Theorem 3, and y satisfy*

$$y^p - y = b, \tag{19}$$

where b is in C. Then $P(y)$ is cyclic of degree p^2 over P if and only if there exists an integer $h \neq 0$ and a quantity λ of P such that

$$hb - g = \lambda \tag{20}$$

is a p-difference of C.

3. **On p-differences of $P(u)$.** Let $C = P(u)$ be cyclic of degree p over P, $u^p = u + \alpha$. We may first prove

THEOREM 6. *Let $a = c^p - c$ where a and c are in P. Then the equation $x^p - x = a$ has the unique solutions*

$$c, c + 1, \cdots, c + p - 1. \tag{21}$$

For if $x^p - x = c^p - c$ then $(x - c)^p = x^p - c^p = x - c$. Let $y = x - c$ so that $y^p - y = 0$. By Theorem 1 this equation has the solutions $0, 1, 2, \cdots, p - 1$.

Let now $a = \alpha_0$ in P. If $\alpha_0 = c^p - c$ where c is in $P(u)$ then either c is in P or, by Theorem 1, the quantity c of $P(u)$ satisfies a normed equation and has the form $c = ku + \gamma$. But then $c^p - c = k\alpha + \gamma^p\gamma$ so that $\alpha_0 - k\alpha$ is a p-difference of P. If c is in P then $\alpha_0 - 0.\alpha$ is a p-difference of P and we have

THEOREM 7. *A quantity α_0 of P is a p-difference of $C = P(u)$ if and only if there exists an integer $k = 0, 1, \cdots, p - 1$ such that $\alpha_0 - k\alpha$ is a p-difference of P.*

THEOREM 8. *The degree of c is either zero of unity when a has degree zero.*

We suppose now that

$$a = \alpha_0 u^t + \cdots + \alpha_t \neq 0 \qquad (0 \leq t < p,\ \alpha_i \text{ in } P) \tag{22}$$

so that

$$a' = a(u+1) - a = \alpha_0' u^{t-1} + \cdots + \alpha_t', \tag{23}$$

where

$$\alpha_0' = t\alpha_0. \tag{24}$$

Then if we define $a^{(k)} = [a^{(k-1)}]'$ we evidently have

$$a^{(t)} = t(t-1)\cdots 2\cdot 1\cdot \alpha_0 \neq 0. \tag{25}$$

Conversely let $a^{(t)} = \beta_0$ be given. Then there exists a unique polynomial $b^{(t-1)}$ of the first degree and zero constant term such that $b^{(t-1)}(u+1) - b^{(t-1)} = a^{(t)}$. In fact $b^{(t-1)} = \beta_0 u$. But then $a^{(t-1)} = b^{(t-1)} + \mu_{t-1}$. Then if $a = c^p - c$ we have $a(u+1) = c(u+1)^p - c(u+1)$ so that $a' = (c')^p - c'$ where $c' = c(u+1) - c$. Evidently the degree of c' is $\sigma - 1$ where σ is the degree of c.

Similarly $a'' = (c'')^p - c''$, $\cdots$, $a^{(t)} = [c^{(t)}]^p - c^{(t)}$. But $a^{(t)} = (t!)\alpha_0 = s\alpha_0$ where s is an integer not divisible by p. There exists an integer r such that $rs = 1$, $r^p = r$,

$$rs\alpha_0 = \alpha_0 = r[(c^{(t)})^p - c^{(t)}] = d^p - d$$

where $d = rc^{(t)}$ has the same degree $\sigma - t$ as $c^{(t)}$. By Theorem 8 the degree of d is zero or unity. If $t = p-1$ then $\sigma \neq t+1$ so that we have

THEOREM 9. *Let a have degree p-1 and leading coefficient α_0. Then a necessary condition that a be a p-difference of C is that α_0 be a p-difference of P.*

THEOREM 10. *Let a have degree $t < p-1$, leading coefficient α_0. Then a necessary condition that a be a p-difference of C is that α_0-kα shall be a p-difference of P.*

Suppose now that the necessary condition of the above theorems is satisfied. We shall then determine necessary and sufficient conditions involving only quantities in P by the following process. We assume that a given quantity $a' = a(u+1) - a$ is a p-difference of C. Then $a' = d^p - d$ and d determines a unique polynomial c such that $c' = d$. But then $a' = (c')^p - c = a(u+1) - a$ has a unique solution apart from an additive constant. It has the obvious solution $a = c^p - c$. Hence, if a is given, we have $a = c^p - c + \mu$ and, if a is also a p-difference of C then $c^p - c + \mu = f^p - f$, μ is a p-difference of C. We have proved

THEOREM 11. *Let a in C have the property that $a(u+1) - a = a' = d^p - d$, where d in C has degree $t < p-1$. Then $d = c'$ where c is uniquely determined and has zero constant term, $a = c^p - c + \mu$, μ in P. The quantity a is then a p-difference of C if and only if μ is a p-difference of P.*

The method of application of the above criteria is as follows. If a has degree $p-1$ then first $(p-1)/\alpha_0 = \beta^p - \beta$, β in P. We then compute $d_1 = \beta\mu$ and find that $a^{(t-1)} = d_1{}^p - d_1 + \mu_1$. Necessarily then $\mu_1 - k\alpha$ is a p-difference of P. This determines a polynomial c_1 of degree zero or unity such that $a^{(t-1)} = c_1{}^p - c_1$. We then determine d_2 such that $a^{(t-2)} = d_2{}^p - d_2 + \mu_2$, and so forth. The case where a has degree $t \leq p-2$ is completed similarly except that β may this time be in $P(u)$ and the final c such that $a = c^p - c$ may have degree $t+1$.

The above process may of course be applied to determine the irreducibility of an equation $x^p - x - b = 0$ in C containing b. We may also revise Theorem 5 to be

THEOREM 12. *The field $P(y)$ of Theorem 5 is cyclic of degree p^2 over P if and only if there exists an integer $h \not\equiv 0$ such that*

$$(hb - g)'$$

is a p-difference of C.

For by Theorem 11 if $(hb-g)' = d^p - d$ then $hb - g = c^p - c + \mu$, μ in P as desired in Theorem 5. The converse is obvious.

4. **Non-cyclic normal fields of degree p^2 over P.** It is well known that every non-cyclic normal field of degree p^2 over P is a direct product of two cyclic fields and conversely every such direct product is non-cyclic and normal. Hence let

$$u^p = u + \alpha, \qquad v^p = v + \beta, \tag{26}$$

and consider the field $P(u, v)$. If $P(u)$ and $P(v)$ are equivalent then $P(u)$ contains a quantity v_0 satisfying $v_0^p = v_0 - \beta$. By Theorem 2 we have $v_0 = -ku + \gamma$, γ in P, $k = 1, 2, \cdots, p-1$, so that $v_0^p - v_0 = -k^p u^p + \gamma^p + ku - \gamma = -k\alpha + \gamma^p - \gamma = \beta$. Conversely if $p + k\alpha$ is a p-difference of P then $-ku + \gamma = v_0$ satisfies $v_0^p = v_0 + \beta$ and $P(u)$ and $P(v)$ are equivalent. Thus we have the corollary to Theorem 2.

THEOREM 13. *The two cyclic fields defined by* (26) *are equivalent if and only if there exists an integer $k \not\equiv 0$ such that $\beta + k\alpha$ is a p-difference of P.*

If $P(u, v)$ is normal of degree p^2 over P then evidently $P(v)$ and $P(u)$ cannot be equivalent. Conversely if v is not in $P(u)$ then $P(u, v)$ is cyclic of degree p over $P(u)$ by Theorem 1, so that $P(u, v)$ has degree p^2 over P and is evidently normal. We have proved

THEOREM 14. *Let $N = P(u, v)$ where* (26) *holds. Then N is a non-cyclic normal field of degree p^2 over P if and only if none of the quantities*

$$\alpha, \beta, \beta + \alpha, \cdots, \beta + (p-1)\alpha \tag{27}$$

are p-differences of P.

Let N be defined as in Theorem 13. If $v_k = v + ku$, $k = 0, 1, \cdots, p-1$, then $v_k^p - v_k = v^p - v + k^p u^p - ku = \beta + k(u + \alpha) - ku = \beta + k\alpha$ is in

P, v_k generates a cyclic field $P(v_k)$ of degree p over P by (27). Moreover the $p+1$ cyclic fields $P(v_k)$, $P(u)$ are non-equivalent.

For let $P(v_k)$ be equivalent to $P(u)$. By Theorem 14 the field $P(v_k) = P(v + ku)$ is evidently cyclic of degree p over $P(u)$. But this contradicts the hypothesis that $P(u)$ and $P(v_k)$ are equivalent. Let then $P(v_k)$ be equivalent to $P(v_t)$, $0 \leq k < t < p$ so that, by Theorem 13,

$$\beta + t\alpha - s(\beta + k\alpha)$$

is a p-difference of P. If $s = 1$ then $t - k \neq 0$, $(t-k)\alpha$ is equal to $\gamma^p - \gamma$, γ in P, and $(t-k)^{p-2}[(t-k)\alpha] = \alpha = [(t-k)\gamma]^p - (t-k)\gamma$ is a p-difference of P contrary to (27). Hence $1 - s = r \neq 0$, $r^{p-2}[r\beta + (t - sk)\alpha] = \beta + r_0\alpha$ is a p-difference of P a contradiction of (27).

Hence $P(u)$, $P(v_k)$ are non-equivalent cyclic fields of degree p over P contained in $P(u, v)$. Conversely $P(u, v)$ has degree p^2 over P if and only if all the fields $P(u)$, $P(v_k)$ have degree p over P, that is they are cyclic. We shall now show that $P(u, v)$ has no further sub-fields besides P, $P(u, v)$, $P(u)$, $P(v_k)$ already determined.

For let $P(z)$ be a sub-field of $P(u, v)$ distinct from P, $P(u, v)$. It is well known[3] that necessarily $P(z)$ is cyclic of degree p over P. Hence $z^p = z + \gamma$, γ in P. But if $P(z) \neq P(u)$ then $P(z, u) = P(v, u)$, that is z is a primitive quantity of $P(v, u)$ of degree p over $P(u)$. By Theorem 2

$$z = tv + a \qquad (t = 1, 2, \cdots, p-1)$$

where a is in $C = P(u)$. Then $z^p - z = t^p v^p + a^p - tv - u = t(v + \beta) - tv + a^p - a = \gamma$ in P. Thus $a^p - a = \gamma - t\beta$. By Theorem 2 we have $a = ru + \epsilon$ where ϵ is in P and $r = 0, 1, \cdots, p-1$. Let $z_0 = (z - \epsilon)t^{p-2} = (tv + ru)t^{p-2} = v + ku$. Evidently $P(z_0) = P(z) = P(v_k)$.

THEOREM 15. *The field N of Theorem 13 has precisely $p+1$ non-equivalent proper sub-fields distinct from P and they are the cyclic fields*

$$P(u),\ P(v_k) \qquad (k = 0, 1, \cdots, p-1) \tag{28}$$

of degree p over P where

$$v_k = v + ku, \qquad v_k^p - v_k = \beta + k\alpha, \qquad u^p - u = \alpha. \tag{29}$$

Conversely N has degree p^2 over P if and only if the fields $P(u)$, $P(v_k)$ defined by (29) are all cyclic of degree p over P.

5. **Non-normal fields.** Let

$$P(x) > P(u) > P \tag{30}$$

where $P(x)$ is cyclic of degree p over $P(u)$ which is cyclic of degree p over P. Then if $C = P(u)$ we have $P(x) = C(x)$,

$$u^p = u + \alpha, \qquad x^p = x + a \tag{31}$$

[3] cf. the fundamental theorem on the galois theory in Hasse, Höhere Algebra II.

with α in P and a in $P(u)$. We now wish conversely to define $P(x)$ by (31), assume that α is not a p-difference of P and determine necessary and sufficient conditions on a that $P(x)$ shall be a non-normal field of degree p^2 over P. By Theorem 1 we have

THEOREM 16. *Equations* (31) *define* $P(x, u)$ *of degree* p^2 *over* P *if and only if* a *is not a* p*-difference of* $P(u)$

Next suppose that $P(x, u) = P(v, u)$, $v^p = v + \beta$, β in P. Then $x = kv + c$ where $k = 1, 2, \cdots, p - 1$ and c is in $P(u)$. Then $x^p - x = k^p v^p + c^p - kv - c = k\beta + c^p - c = a$. Thus $a - k$ is a p-difference of C. But $\delta = k\beta$ is in P and hence $a - \delta$ is a p-difference of C if and only if $a' = a(u + 1) - a$ is a p-difference of C by the proof of Theorem 11. We have.

THEOREM 17. *Let* $C = P(u)$ *be cyclic of degree* p *over* P, $x^p = x + a$ *where* a *is in* C, a *is not a* p*-difference of* C. *Then* $P(x, u)$ *is a non-cyclic normal field of degree* p^2 *over* P *if and only if* a' *is a* p*-difference of* C.

We now apply Theorems 12, 17 to obtain.

THEOREM 18. *Let* $C = P(u)$ *be cyclic of degree* p *over* P, $x^p = x + a$, a *in* C. *Then* $P(x)$ *is a non-normal field of degree* p^2 *over* P *if and only if*

$$(a - hg)' \qquad (h = 0, 1, \cdots, p - 1)$$

is not a p*-difference of* C *where* g *is the uniquely defined polynomial of Theorem* 3.

For evidently if a' is not a p-difference of C then a is also not a p-difference of C so that $P(x, u)$ has degree p^2. By Theorem 17 $P(x, u)$ is not a non-cyclic normal field so that $P(x, u) = P(x)$. By Theorem 12 $P(x)$ is non-cyclic and hence non-normal. The converse is obvious.

We pass now to the case $P = 2$.

6. **On quartic fields.** The only quartic separable fields with quadratic sub-fields are of course fields $P(x) > P(u) > P$ where $P(u)$ is cyclic of degree 2 over P, $P(x)$ is cyclic of degree 2 over $P(u)$. Hence let

$$u^2 = u + \alpha, \qquad x^2 = x + a, \qquad a = \beta u + \gamma, \tag{32}$$

where in particular α is not a 2-difference of P.

We first compute g. We wish $g = \gamma_0 u$, $g(u + 1) - g(u) = \gamma_0(u + d) - \gamma_0 u = \gamma_0 = (u + \alpha) - u = \alpha$. Hence

$$g = \alpha u. \tag{33}$$

As an immediate application of Theorem 3 we have

THEOREM 19. *Every cyclic quartic field over* P *of characteristic* 2 *is generated by*

$$x^2 = x + \alpha u + \gamma, \qquad u^2 = u + \alpha, \qquad \alpha, \gamma \text{ in } P, \tag{34}$$

such that $\alpha \neq \delta^2 - \delta$ *for any* δ *of* P.

We also have immediately

THEOREM 20. *Every non-cyclic normal quartic field over* P *is a field* $P(u, v)$,

$$u^2 = u + \alpha, \qquad v^2 = v + \beta \tag{35}$$

where

$$\alpha, \beta, \alpha + \beta \tag{36}$$

have not the form $\delta^2 - \delta$ *for any* δ *of* P. *The only proper sub-fields of* $P(u, v)$ *distinct from* P *are the fields*

$$P(u), P(v), P(w),$$

where

$$w = u + v, \qquad w^2 = w + \alpha + \beta.$$

We finally wish to determine a so that $P(x)$ shall be non-normal. We thus wish $(a - hg)' = (\beta u + \gamma - h\alpha u)' = (\beta - h\alpha)(u + 1) + \gamma - [(\beta - h\alpha)u + \gamma] = \beta - h\alpha$ not a 2-difference of P. Since $h = 0, 1$ and $-1 = 1$ in P we have

THEOREM 21. *Let* $u^2 = u + \alpha$, $x^2 = x + \beta u + \gamma$ *with* α, β, γ *in* P. *Then* $P(x)$ *is a non-normal quartic field over* P *if and only if*

$$\alpha, \beta, \alpha + \beta \tag{37}$$

are not 2-differences of P.

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CONCERNING POWERS OF CERTAIN CLASSES OF IDEALS IN A CYCLOTOMIC REALM WHICH GIVE THE PRINCIPAL CLASS

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1. Introduction

It is a familiar theorem that, if $q^f \equiv 1$, mod p, where q and p are rational primes, and f is the smallest exponent for which this congruence holds, then, in the cyclotomic number-realm $k(\zeta = e^{2\pi i/p})$,

$$[q] = \prod_{j=1}^{e} [q, F_j(\zeta)], \qquad p - 1 = ef.^1$$

The ideals $[q, F_j(\zeta)]$ remain unchanged by the substitutions: S^{ke}, $k = 0, 1, \cdots, f-1$, where S^i is (ζ/ζ^{γ^i}), and are prime ideals of the first degree in the sub-realm of $k(\zeta)$ generated by the e periods:

$$\eta_i = \sum_{j=0}^{f-1} \zeta_{je+i}, \qquad i = 0, 1, \cdots, e-1,$$

γ being a primitive root of p, and $\zeta_h = \zeta^{\gamma^h}$.[2] If, in particular, $f = 1$, we have

$$[q] = \prod_{i=0}^{p-2} \mathfrak{q}_i,$$

where $\mathfrak{q} = \mathfrak{q}_0 = [q, g^k - \zeta]$, g being a primitive root of q, $q - 1 = kp$, and $S^i\mathfrak{q} = \mathfrak{q}_i$.[3]

Concerning the ideals $\mathfrak{q}_i$ mentioned above it is known that:

$$[(\zeta, \eta)^p] = \prod_{i=0}^{p-2} \mathfrak{q}_i^{\gamma^{-i+(p-1)/2}},$$

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[1] D. Hilbert, *Gesammelte Abhandlungen*, V. 1, Berlin, 1932. Compare articles 93, 96, 97, and 99. R. Fricke, *Lehrbuch der Algebra*, V. 3, 1928, p. 193. Compare also pp. 196, 198.

[2] For a generalization of the "Kreistheilungsperioden" in the composite case, see L. Fuchs, Journal für Mathematik, V. 61, 1862, H. Weber, *Lehrbuch der Algebra*, Ed. 2, V. 2, 1899, Article 22.

[3] E. Landau, *Vorlesungen über Zahlentheorie*, V. 3, 1927, Theorem 1019, pp. 289–90. For a more general theorem compare R. Fricke, *loc. cit.*, p. 210.

where γ_j is the least positive residue of γ^j, mod p, $\eta = e^{2\pi i/q}$, and (ζ, η) is the Lagrange Resolvent function:

$$\sum_{t=1}^{q-1} \zeta^{\text{ind}\, t} \eta^t,$$

ind t being taken with respect to g.[4] Another result is:

$$\prod_{i=0}^{p-2} \mathfrak{q}_i^{m_i} \sim \mathfrak{o},$$

where $\mathfrak{o}$ is the unit ideal, the symbol denotes equivalence in the broader sense, and the exponents m_i are given by:

$$m_i = (\gamma_{-i+\text{ind}\, a} + \gamma_{-i+\text{ind}\, b} - \gamma_{-i+\text{ind}\,(a+b)})/p,$$

the index being taken with respect to γ, a and b being any rational integers such that: $ab(a+b) \not\equiv 0$, mod p.[5] A theorem somewhat similar to the two given above is the following: If C is an arbitrary ideal-class of $k(\zeta)$ and J an ideal in C, and $SC, S^2C, \cdots, S^{p-2}C$ denote the ideal classes of $SJ, S^2J, \cdots, S^{p-2}J$, then

$$C^{\sum_{i=0}^{p-2} q_{-i} S^i} = 1,$$

where $q_{-i} = (\gamma\gamma_{-i} - \gamma_{-i+1})/p$, and $(C)^{q_{-i}S^i} = (S^iC)^{q_{-i}}$, following the "symbolic powers" of Hilbert.[6] H. H. Mitchell[7] has obtained a result which may be regarded as a further extension of the results already mentioned. It may be stated as follows: If $q^{p-1/2e} \equiv 1$, mod p, where $(p-1)/2e$ is an odd integer other than unity, then it is known that, in the realm of degree $2e$ determined by the $2e$ periods formed from pth roots of unity, $[q]$ is the product of $2e$ conjugate prime ideals, which form e conjugate imaginary pairs. Selecting one ideal from each of the e conjugate imaginary pairs, the product of these raised to the power h, where h denotes the "first-factor" of the class number of the realm,[8] is a principal ideal. The main purpose of this paper is to extend this theorem to the Cyclotomic Realm generated by a primitive p^sth root of unity, p any rational

[4] E. Landau, *loc. cit.*, Theorem 1020, p. 292. Compare also E. Kummer, *Theorie der idealen Primfaktoren der komplexen Zahlen u.s.w.*, Abhandlungen der Berliner Akad. von 1856.

[5] H. H. Mitchell: Transactions of the American Mathematical Society, 1916, V. 17, 2, p. 173 (replacing i *by* γ^{-i}). Compare also E. Kummer, Journal für Mathematik, V. 44, 1851, pp. 106–121. A. Lévy and Th. Got: *Théorie des Corps de Nombres Algèbriques*, 1913, Note VI by Got. Here the author discusses researches made by Kummer and others on Fermat's last Theorem.

[6] D. Hilbert, *loc. cit.*, Article 109. See also E. Kummer, Journal für Mathematik, V. 35, 1847; Journal de Mathématiques, V. 16, 1851. These results are also given in the Bulletin of the National Research Council, Number 62, *Algebraic Numbers*-11, 1928.

[7] H. H. Mitchell, Transactions of the American Mathematical Society, 1918, V. 19, 2, pp. 119–126.

[8] E. Kummer: Journal für Mathematik, V. 40, 1850, p. 112. For a more general result, compare L. Fuchs, *Ibid*, V. 65, 1866.

prime. In so doing, sets of exponents are obtained for the general cyclotomic realm, $k(e^{2\pi i/n})$, n composite, from which the above theorem is deduced for $n = p^s$. In concluding this introduction, I would like to express my deep appreciation to Professor H. H. Mitchell, without whose valuable suggestions and kindly advice this paper would certainly never have been written.

2. The Ψ-Function. Derived Sets of Exponents

We introduce the Jacobi-Kummer Cyclotomic Ψ-function, as generalized by Mitchell:[9]

$$\Psi_{-a,\,-b}(\alpha) = \sum_{\tau} \alpha^{-b \text{ ind } \tau+(a+b) \text{ ind } (\tau+1)},$$

where $\alpha = e^{2\pi i/n}$, $n = 2^\lambda p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, a and b are rational integers such that: $a \not\equiv 0, b \not\equiv 0, (a+b) \not\equiv 0$, mod n, τ ranges over all the marks of a Galois field of order q^t except 0 and -1 if q is odd, and 0 and 1 if q is 2, ind τ being taken with respect to a primitive root in the Galois field, and t is any exponent for which $q^t \equiv 1$, mod n. This function has two characteristic properties, viz:

$$\text{(a) } \Psi(\alpha)\Psi(\alpha^{-1}) = q^t \qquad \text{(b) } \Psi(\alpha^q) = \Psi(\alpha).$$

Taking $t = 1$, we have:

$$\Psi_{-a,\,-b}(\alpha) = \sum_{\tau=1}^{q-2} \alpha^{-b \text{ ind } \tau+(a+b) \text{ ind } (\tau+1)}.$$

Further:

$$[\Psi_{-a,\,-b}(\alpha)] = \prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{m_{a_k}}, \tag{1}$$

where $a_1, a_2, \cdots, a_{\varphi(n)}$ are the $\varphi(n)$ totitives of n, $\mathfrak{q}(\alpha)$ is a prime ideal factor of q, $\mathfrak{q}_{a_k} = \mathfrak{q}(\alpha^{a_k^{-1}})$, ind τ is taken with respect to a primitive root g of q chosen so that: $g^{q-1/n} \equiv \alpha$, mod $\mathfrak{q}(\alpha)$, and the exponents

$$m_{a_k} = (|\, aa_k \,| + |\, ba_k \,| - |\, (a+b)a_k \,|)/n,$$

$|\, x \,|$ denoting the least positive residue of x, mod n. The ideals $\mathfrak{q}_{a_k}$ are prime ideals of the first degree in the cyclotomic realm $k(\alpha)$, and we have:

$$[q] = \prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}.^{10}$$

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[9] H. H. Mitchell, *loc. cit.* (1916), pp. 165–177. For the Jacobi function, compare H. Weber, *loc. cit.*, V. 1, Articles 177, 178; V. 2, Article 203. P. Bachmann, *Die Lehre von der Kreisteilung*, Achte Vorlesung.

[10] H. H. Mitchell, *loc. cit.*, p. 173.

By combining these exponents m_{a_k} in various ways it is possible to obtain new sets of exponents that may be of interest. For example, holding a fixed and letting b take all possible values between 0 and n, we have from relation (1):

$$\prod_{b=1}^{n-1}{}^{(n-a)}\,[\Psi_{-a,-t}(\alpha)] = \prod_{b=1}^{n-1}{}^{(n-a)}\prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{m_{a_k,b}} = \prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{\sum_{b=1}^{n-1}{}^{(n-a)}\,m_{a_k,b}},$$

where the super-script in the product and summation signs denotes what values are to be omitted, and we have written $m_{a_k,b}$ for m_{a_k} above. But, making use of the fact that $|\,aa_s\,| + |\,-aa_s\,| = n$, we have:

$$\sum_{b=1}^{n-1}{}^{(n-a)}\,m_{a_k,b} + 1 = \sum_{b=0}^{n-1} m_{a_k,b} = \Big(n\,|\,aa_k\,| + \sum_{b=0}^{n-1}(|\,ba_k\,| - |\,(a+b)a_k\,|)\Big)\Big/ n = |\,aa_k\,|,$$

where $|\,x\,| = 0$ if $n \mid x$. Remembering that $\prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{r} \sim \mathfrak{o}$, where r is any integer, we are thus led to the result:

$$\prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{n_{a_k}} \sim \mathfrak{o}, \tag{1.1}$$

where $n_{a_k} = |\,aa_k\,| + K$, K being any integer.

Another interesting way of combining the exponents in (1) is to consider $b \equiv 1$, mod n, and let a take the values $1, 2, \cdots, s-1$, where $1 < s < n$. We then have:

$$\prod_{a=1}^{s-1}\prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{m_{a_k,a}} = \prod_{a=1}^{s-1}[\Psi_{-a,-1}(\alpha)] = \prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{\sum_{a=1}^{s-1} m_{a_k,a}},$$

where, as above, we have written $m_{a_k,a}$ for m_{a_k}. But:

$$\sum_{a=1}^{s-1} m_{a_k,a} = (s\,|\,a_k\,| - |\,sa_k\,|)/n = t_{a_k},$$

say. We now have:

$$\prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{t_{a_k}} \sim \mathfrak{o},$$

and we can replace s in the formula for t_{a_k} by $s + \kappa n = M$, where κ is any integer. For, if $T_{a_k} = (M\,|\,a_k\,| - |\,Ma_k\,|)/n$, then:

$$\prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{T_{a_k}} = \prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{t_{a_k}+\kappa|a_k|} = \prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{t_{a_k}}\prod_{k=1}^{\varphi(n)}\mathfrak{q}_{a_k}^{\kappa|a_k|} \sim \mathfrak{o},$$

since $\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{|a_k|} \sim \mathfrak{o}$ by virtue of (1.1) if we take $K = 0$ and $a \equiv 1$, mod n. We have then:

$$\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{S_{a_k}} \sim \mathfrak{o}, \tag{1.2}$$

where $S_{a_k} = (M \mid a_k \mid - \mid Ma_k \mid + nr)/n$, r being any integer, and M is of the form: $t + \kappa n$, $1 < t < n$, κ being any integer. It is interesting to note that, for $n = p$, an odd prime, taking $a_k = \gamma^{-i}$, and choosing $K = 0$ and $a \equiv -1$, mod n in (1.1) and $s = \gamma$ in the exponents t_{a_k}, we obtain results mentioned in the introduction to this paper.[11]

3. The Group of $\varphi(n)$ Totitives and Their "Characters"[12]

We consider the Abelian group formed by the $\varphi(n)$ totitives of n, $a_1, a_2, \cdots, a_{\varphi(n)}$, the rule of combination being multiplication. Taking the elements a_i as positive and arranged so that $a_i > a_j$ if $i > j$, we have:

$$a_{\varphi(n)-k} \equiv a_{\varphi(n)}\, a_{k+1}, \text{ mod } n. \tag{2}$$

A basis of this Abelian group is $C_{-1}, C_0, \cdots, C_m$ defined as follows for $n = 2^\lambda p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$:

(a). $\lambda = 0$ or 1. $C_{-1} = C_0 = 1$, and C_i satisfies the m simultaneous congruences:

$$C_i \equiv g_i, \text{ mod } p_i^{k_i};\ C_i \equiv 1, \text{ mod } p_j^{k_j},\ i \neq j; \qquad i, j = 1, 2, \cdots, m,$$

where g_i is a primitive root of p_i such that: $g_i^{p_i - 1} \not\equiv 1$, mod p_i^2, when $k_i > 1$.

(b). $\lambda = 2$. $C_{-1} = 1$; $C_0 \equiv -1$, mod 2^λ; $C_0 \equiv 1$, mod $p_i^{k_i}$, $i = 1, 2, \cdots, m$; the C_i satisfy the $(m + 1)$ simultaneous congruences:

$$C_i \equiv 1, \text{ mod } 2^\lambda;\ C_i \equiv 1, \text{ mod } p_j^{k_j};\ C_i \equiv g_i, \text{ mod } p_i^{k_i}.$$

(c). $\lambda > 2$. $C_{-1} \equiv -1$, mod 2^λ; $C_{-1} \equiv 1$, mod $p_i^{k_i}$.
$C_0 \equiv 5$, mod 2^λ; $C_0 \equiv 1$, mod $p_i^{k_i}$.
$C_i \equiv 1$, mod 2^λ; $C_i \equiv 1$, mod $p_j^{k_j}$;
$C_i \equiv g_i$, mod $p_i^{k_i}$.

We remark that if, under (a),

$$a_k \equiv \prod_{i=1}^{m} C_i^{\gamma_i}, \text{ mod } n,$$

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[11] For more general results compare R. Fricke, *loc. cit.*, p. 215. L. Stickelberger: Mathematische Annalen, V. 37, 1890, pp. 321–367; Encyclopédie des Sciences Mathématiques, t.I, V. 3, pp. 440–2.

[12] H. Weber, *loc. cit.*, V. 2, Articles 16, 17, 18, pp. 60–68.

then $\gamma_i = \text{ind}_{g_i}^{p_i^{k_i}} a_k$, the super-script[13] denoting the modulus, so that, for all elements of the group, $\gamma_i = 0, 1, \cdots, \varphi(p_i^{k_i}) - 1$. Under (b), if

$$a_k \equiv \prod_{i=0}^{m} C_i^{\gamma_i}, \text{ mod } n,$$

then $a_k \equiv (-1)^{\gamma_0}$, mod 2^λ, so that $\gamma_0 = 0, 1$. Finally, under (c), if:

$$a_k \equiv (C_{-1})^{\gamma_{-1}} \prod_{i=0}^{m} C_i^{\gamma_i}, \text{ mod } n,$$

then: $a_k \equiv (-1)^{\gamma_{-1}}\ (5)^{\gamma_0}$, mod 2^λ, so that: $\gamma_{-1} = 0, 1$; $\gamma_0 = 0, 1, \cdots, 2^{\lambda-2} - 1$.

Preserving the same set of indices for a_k as given above under each of the three cases, the $\varphi(n)$ "characters" of a_k are given by:

$$\text{(a')}\ \chi_b(a_k) = \prod_{i=1}^{m} \epsilon_i^{\gamma_i \beta_i}, \text{ where } \epsilon_j = e^{2\pi i/\varphi(p_j^{k_j})}.$$

$$\text{(b')}\ \chi_b(a_k) = (-1)^{\gamma_0 \beta_0} \prod_{i=1}^{m} \epsilon_i^{\gamma_i \beta_i}.$$

$$\text{(c')}\ \chi_b(a_k) = (-1)^{\gamma_{-1}\beta_{-1}} (\epsilon_0)^{\gamma_0 \beta_0} \prod_{i=1}^{m} \epsilon_i^{\gamma_i \beta_i}, \text{ where } \epsilon_0 = e^{2\pi i/2^{\lambda-2}}.$$

In each case the β's represent the indices of b, and the $\varphi(n)$ "characters" are obtained by allowing $\beta_{-1}, \beta_0, \beta_1, \cdots, \beta_m$ to take the set of values given for γ_{-1}, $\gamma_0, \gamma_1, \cdots, \gamma_m$ in the last paragraph, depending on the case considered.[14]

4. The "First-Factor" of the Class Number[15]

In order to preserve the continuity of this paper, it seems advisable at this point to introduce the finite expression for the "first-factor" of the Class Number

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[13] This will be omitted hereafter.

[14] H. Weber, *loc. cit.*, V. 2, p. 68.

[15] M. Gut: Commentarii Math. Helvetici, V. 1, 1929, pp. 160–226. In particular, the reader is advised to refer to Articles 1, 2, 3, 5, and 11 before proceeding with this section. N. G. W. H. Beeger: Akad. v. Wetenschappen te Amsterdam, 6 papers, V. XXI, XXII, XXIII, 1919–1922. D. Hilbert, *loc. cit.*, Articles 116, 117. E. Kummer: Ber. der K. Akad. der Wiss. zu Berlin, 1861; ibid, 1863. In the first of these papers, Kummer lists the "first-factors" for $n = 1 - 100$. L. Kronecker: *Oeuvres*, V. 1, p. 123, 1863. In this paper the author proves that twice the "first" or "second" factors is always a rational integer. For the case of an odd prime and the power of a prime compare H. Weber, *loc. cit.*, V. 2, Articles 214–223. Here, for $n = 2^\lambda$, Weber proves that the class number is an odd integer. A. Lévy and Th. Got, *loc. cit.*, Note V by Got. See the correction in Beeger, *loc. cit.*, p. 414, and also in the Bulletin of the National Research Council, V. 5, Part 3, February, 1923, *Algebraic Numbers*. P. Dirichlet and R. Dedekind: *Vorlesungen über Zahlentheorie*, Ed. 4, 1894, Supplement XI. Article 185. R. Fueter: *Synthetische Zahlentheorie*, Ed. 2, 1925, p. 223. Absolute values are used here to make the factors of the class number positive.

in the general Cyclotomic number-realm as given by Hilbert. For brevity, we will confine our discussion to the case $\lambda = 0$, it being immediately extensible to the general case. Writing $\varphi(n) = 2\nu$ and using Hilbert's notation, the "first-factor" is an expression of the form:

$$H = \prod_{(\beta_1, \beta_2, \cdots)} \sum_{K=1}^{n-1} \left[\overbrace{\beta_1, \beta_2, \cdots}^{K}\right] K \div (2n)^{\nu-1},$$

the product being taken for $\sum_{i=1}^{m} \beta_i$ odd. Now:

$$\left[\overbrace{\beta_1, \beta_2, \cdots}^{K}\right] = [K \mid p_1^{k_1}]^{\beta_1} [K \mid p_2^{k_2}]^{\beta_2} \cdots,$$

where $[K \mid p_j^{k_j}] = \epsilon_j^{\operatorname{ind} g_j^K}$ or 0 (see section 3) according as $p_j \nmid K$ or $p_j \mid K$. Further, in all cases, $[K \mid p_j^{k_j}]^0 = 1$. Suppose we consider the $2^m - 1$ sets of factors in the numerator of H obtained by letting first none of the β's be 0, then letting them be 0 one at a time, two at a time, $\cdots$, $(m-1)$ at a time. For clarity, let us consider a factor of H when, say, $\beta_1 = \beta_2 = \cdots = \beta_s = 0$, but $\beta_i \neq 0$ for $i > s$. The factors occurring in H then assume the form:

$$\sum_{K=1}^{n-1} \left[\overbrace{\beta_{s+1}, \beta_{s+2}, \cdots, \beta_m}^{K}\right] K = \sum_{k=0}^{r_i-1} \sum_{A} \left[\overbrace{\beta_{s+1}, \cdots, \beta_m}^{A+k\,s_i}\right] (A + ks_i),$$

where $r_i = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$, $s_i = p_{s+1}^{k_{s+1}} p_{s+2}^{k_{s+2}} \cdots p_m^{k_m}$, and A goes over the positive numbers less than and prime to s_i. Since, however, $A + ks_i \equiv A$, mod $p_t^{k_t}$, where p_t is any prime dividing s_i, we may write the above factor:

$$\sum_{k=0}^{r_i-1} \sum_{A} \left[\overbrace{\beta_{s+1}, \cdots, \beta_m}^{A}\right] (A + ks_i) = \sum_{k=0}^{r_i-1} \sum_{A} \left(\prod_{j=s+1}^{m} \epsilon_j^{\operatorname{ind} g_j^{A} \cdot \beta_j} \right) (A + ks_i) =$$

$$\sum_{k=0}^{r_i-1} \sum_{A} \chi_b^{s_i}(A)\, [A + ks_i],$$

where $\chi_b^{s_i}(A)$ denotes a "group character" of A, the elements of the group being the numbers prime to s_i, and the set of indices for b being $\beta_{s+1}, \beta_{s+2}, \cdots, \beta_m$ (compare section 3).

We observe that:

$$\sum_{k=0}^{r_i-1} \sum_{A} \chi_b^{s_i}(A)\, [A + ks_i] = \sum_{k=0}^{r_i-1} \left[\sum_{A} \chi_b^{s_i}(A) \cdot A + ks_i \sum_{A} \chi_b^{s_i}(A) \right] =$$

$$r_i \sum_{A} \chi_b^{s_i}(A) \cdot A,$$

since $\sum_{A} \chi_b^{s_i}(A) = 0, b \not\equiv 1, \text{mod } s_i$.[16] If we understand by (x, y) the greatest common divisor of x, y, we may sum up the above observations in the following

[16] H. Weber, *loc. cit.*, V. 2, p. 53.

THEOREM: *If* $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, $p_1, p_2, \cdots, p_m$ *being distinct odd primes, then in the cyclotomic realm* $k(e^{2\pi i/n})$ *the "first-factor" of the class number is given by the expression:*

$$H = \prod_{s_i} \prod_{b} \left[r_i \sum_{A} \chi_b^{s_i}(A) \cdot A \right] \div (2n)^{r-1},$$

where the outer product is taken over the $2^m - 1$ *divisors* $s_i > 1$ *of* n *such that* $r_i s_i = n$, $(r_i, s_i) = 1$, *the inner product is taken over the elements* b *of the group of* $\varphi(s_i)$ *totitives for which* $\Sigma\beta_k$ *is odd,* $\beta_k \not\equiv 0$, β_k *being the set of indices for* b, *and the summation goes over all positive numbers* A *less than and prime to* s_i. *The expression given for* H *is unaltered if* $n = 2p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, *but if* $n = 2^\lambda p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, $\lambda > 1$, *although we obtain an analogous expression in the numerator, the denominator must be multiplied by* 2.

5. A Generality Due to a Common Property of the Exponents

In order to make the purpose of this section a little clearer to the reader, we remark that if $Q^A \bar{Q}^B \sim 0$, where Q and $\bar{Q}$ are relatively prime, conjugate imaginary ideal factors of $[q]$ such that $[q] = Q\bar{Q}$, then $(Q)^{A-B}(Q\bar{Q})^B \sim 0$, whence $(Q)^{A-B} \sim 0$. With this in mind, we now return to the series of exponents given in section 2, and remark that for those in (1) we have:

$$m_{a_k} + m_{-a_k} = 1;$$

for those in (1.1),

$$n_{a_k} + n_{-a_k} = n + 2K;$$

and for those in (1.2):

$$S_{a_k} + S_{-a_k} = M - 1 + 2r.$$

Suppose, now,

$$\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{L_{a_k}} = [F(\alpha)], \tag{3}$$

the $\mathfrak{q}_{a_k}$ being defined as heretofore, and the L_{a_k} being such that:

$$L_{a_k} + L_{-a_k} = R,$$

where R is independent of the left-hand subscripts, and $L_x = L_y$ if $x \equiv y$, mod n. $F(\alpha)$ is an integer of $k(e^{2\pi i/n})$, so that:

$$\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{L_{a_k}} \sim 0.$$

Applying the substitution: $S^j(\alpha \mid \alpha^{a_j^{-1}})$ in (3) we obtain:

$$\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{L_{a_k a_j^{-1}}} \sim 0, \tag{3.1}$$

and, therefore:

$$\left(\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{L_{a_k a_j^{-1}}} \right)^{l_j} \sim 0, \tag{3.2}$$

where l_j is any rational integer. From (3.2), forming the product for $j = 1, 2, \cdots, \nu$, we obtain:

$$\prod_{j=1}^{\nu}\left(\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{L_{a_k a_j^{-1}}}\right)^{l_j} = \prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{\sum_{j=1}^{\nu} l_j L_{a_k a_j^{-1}}} \sim \mathfrak{o}. \tag{3.3}$$

If we equate the exponents of $\mathfrak{q}_{a_k}$ in (3.3) for $k = 1, 2, \cdots, \nu$, we obtain the $\nu - 1$ equations:

$$\sum_{j=1}^{\nu} l_j (L_{a_k a_j^{-1}} - L_{a_1 a_j^{-1}}) = 0, \qquad k = 2, 3, \cdots, \nu. \tag{4}$$

Equating the exponents of $\mathfrak{q}_{a_k}$ for $k = \nu + 1, \cdots, \varphi(n)$, we obtain the $\nu - 1$ equations:

$$\sum_{j=1}^{\nu} l_j (L_{-a_k a_j^{-1}} - L_{-a_1 a_j^{-1}}) = 0, \qquad k = 2, 3, \cdots, \nu, \tag{4.1}$$

in virtue of:

$$L_{-a_{k+1}} = L_{a_{\varphi(n)-k}}$$

from (2). Since:

$$L_{a_k} + L_{-a_k} = R; \qquad L_{a_k} - L_{-a_k} = 2L_{a_k} - R = d_{a_k},$$

say, then equations (4) and (4.1) are equivalent to:

$$\sum_{j=1}^{\nu} l_j (d_{a_k a_j^{-1}} - d_{a_1 a_j^{-1}}) = 0, \qquad k = 2, 3, \cdots, \nu. \tag{5}$$

We remark that $2 \mid (d_{a_k} - d_{a_j})$ for any a_k, a_j. We now have:

$$\left(\prod_{k=1}^{\nu} \mathfrak{q}_{a_k}\right)^{\sum_{j=1}^{\nu} l_j \cdot L_{a_1 a_j^{-1}}} \cdot \left(\prod_{k=\nu+1}^{\varphi(n)} \mathfrak{q}_{a_k}\right)^{\sum_{j=1}^{\nu} l_j \cdot L_{-a_1 a_j^{-1}}} \sim \mathfrak{o},$$

whence:

$$\left(\prod_{k=1}^{\nu} \mathfrak{q}_{a_k}\right)^{\sum_{j=1}^{\nu} l_j d_{a_j^{-1}}} \sim \mathfrak{o},$$

since $a_1 \equiv 1$, mod n and

$$\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k} = [q].$$

Equations (5) will be satisfied if we take for l_j the values $D_j/2^{\nu-1}$, where D_j is the co-factor of $d_{a_j^{-1}}$ in the first row of:

$$D = \begin{vmatrix} d_{a_1 a_1^{-1}} & d_{a_1 a_2^{-1}} & \cdots & d_{a_1 a_\nu^{-1}} \\ d_{a_2 a_1^{-1}} - d_{a_1 a_1^{-1}} & d_{a_2 a_2^{-1}} - d_{a_1 a_2^{-1}} & \cdots & d_{a_2 a_\nu^{-1}} - d_{a_1 a_\nu^{-1}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ d_{a_\nu a_1^{-1}} - d_{a_1 a_1^{-1}} & \cdots & & d_{a_\nu a_\nu^{-1}} - d_{a_1 a_\nu^{-1}} \end{vmatrix}.$$

Then:

$$\sum_{j=1}^{\nu}{}' l_j d_{a_j^{-1}} = \sum_{j=1}^{\nu}{}' d_{a_j^{-1}} D_j/2^{\nu-1} = D/2^{\nu-1}.$$

By adding the first row to each of the others, we can write D as follows:

$$D = |d_{a_i a_j^{-1}}|, \qquad i, j = 1, 2, \cdots, \nu.$$

6. The Regular Group-Determinant[17]

The determinant D of the last section leads to a consideration of the regular group-determinant:

$$|x_{a_i a_j^{-1}}|, \qquad i, j = 1, 2, \cdots, \varphi(n)$$

in the $\varphi(n)$ independent variables x_{a_k}, $k = 1, 2, \cdots, \varphi(n)$, in which $x_{a_l} = x_{a_m a_r}$ if $a_l \equiv a_m a_r \bmod n$. The substitutions by means of which each row may be obtained from the first row, including identity, constitute a regular substitution group on $\varphi(n)$ letters simply isomorphic with $G = a_1, a_2, \cdots, a_{\varphi(n)}$. This determinant has some remarkable properties, one of which we will make use of in this paper. Since:

$$a_{k+1} a_{\varphi(n)-j}^{-1} \equiv a_{\varphi(n)-k} a_{j+1}^{-1} \bmod n$$

from (2), this group-determinant is centro-symmetric,[18] and therefore equals:

$$|x_{a_i a_j^{-1}} + x_{a_{\varphi(n)-i+1} a_j^{-1}}| \cdot |x_{a_i a_j^{-1}} - x_{a_{\varphi(n)-i+1} a_j^{-1}}|, \qquad i, j = 1, 2, \cdots, \nu.$$

Again, from (2), we have:

$$|x_{a_i a_j^{-1}} \pm x_{a_{\varphi(n)-i+1} a_j^{-1}}| = |x_{a_i a_j^{-1}} \pm x_{a_{\varphi(n)} a_i a_j^{-1}}| =$$

$$F(x_{a_i} + x_{a_{\varphi(n)} a_i}) \text{ or } F'(x_{a_i} - x_{a_{\varphi(n)} a_i}), \qquad i, j = 1, 2, \cdots, \nu,$$

where F and F' are rational, integral, homogeneous functions of degree ν.

On the other hand, we have:

$$|x_{a_i a_j^{-1}}| = \prod_{b} \sum_{t=1}^{\varphi(n)} \chi_b(a_t) x_{a_t}, \qquad i, j = 1, 2, \cdots, \varphi(n),^{19}$$

the product being taken over the $\varphi(n)$ "characters" of the group of $\varphi(n)$ totitives. We consider three cases as heretofore, remarking that, in all cases, $\chi_b(a_1) = 1$:

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[17] The pioneer work in this field was done by G. Frobenius, Sitzungsberichte der K. Preuss. Akad. der Wiss. zu Berlin, 1896–1903; *Oeuvres*, pp. 1363, 1368, 1375. L. E. Dickson has given an elementary exposition of Frobenius' theory in the Annals of Mathematics, 4, 1902, pp. 25–49. For further developments in the theory compare W. Burnside, Proceedings of the London Mathematical Society, V. 29, 1898, pp. 207–224, 546–565; *Theory of Groups*, Ed. 2, 1911, pp. 243–371, 464–84, 499; L. E. Dickson, Transactions of the American Mathematical Society, V. 3, 1902, pp. 285–301; Bulletin of the American Mathematical Society, V. 9, 1902, pp. 394–401; *Modern Algebraic Theories*, 1926, Chapter XIV; H. Weber, *loc. cit.*, V. 2, Ed. 2, pp. 193–218; F. Schur, Berlin. Berichte, 1905, pp. 406–32; H. F. Blichfeldt, *Finite Collineation Groups*, 1917, pp. 116–38.

[18] T. Muir: *Theory of Determinants*, 1882, Article 137.

[19] See, for example, Dickson's elementary exposition of Frobenius' theory or Weber's treatment referred to in foot-note (17).

(A). $\lambda = 0$ or 1: In this case

$$\chi_b(a_{\varphi(n)}) = (-1)^{\sum_{i=1}^{m}\beta_i},$$

β_i being the set of indices for the element b. Therefore,

$$\prod_b \sum_{t=1}^{\varphi(n)} \chi_b(a_t)x_{a_t} = F \text{ or } F',$$

above, according as $\sum_{i=1}^{m} \beta_i$ is even or odd.

(B). $\lambda = 2$: Here

$$\chi_b(a_{\varphi(n)}) = (-1)^{\sum_{i=0}^{m}\beta_i},$$

and, therefore, under this case:

$$\prod_b \sum_{t=1}^{\varphi(n)} \chi_b(a_t)x_{a_t} = F \text{ or } F'$$

according as $\sum_{k=0}^{m} \beta_k$ is even or odd.

(C). $\lambda > 2$: Here, we have:

$$\chi_b(a_{\varphi(n)}) = (-1)^{\beta_{-1}+\sum_{i=1}^{m}\beta_i},$$

so that:

$$\prod_b \sum_{t=1}^{\varphi(n)} \chi_b(a_t)x_{a_t} = F \text{ or } F'$$

according as $\beta_{-1} + \sum_{i=1}^{m} \beta_i$ is even or odd.

Now, with the conditions noted under A, B, and C above, let

$$\prod_b \sum_{t=1}^{\varphi(n)} \chi_b(a_t)x_{a_t} = F' = |x_{a_i a_j^{-1}} - x_{a_{\varphi(n)} a_i a_j^{-1}}|, \qquad i, j = 1, 2, \cdots, \nu.$$

Then, substituting L_{a_t} for x_{a_t}, we have:

$$\prod_b \sum_{t=1}^{\varphi(n)} \chi_b(a_t)L_{a_t} = |L_{a_i a_j^{-1}} - L_{a_{\varphi(n)} a_i a_j^{-1}}| = |d_{a_i a_j^{-1}}| = D, \qquad i, j = 1, 2, \cdots, \nu.$$

Taking account of the condition on the β's and the definition of the "characters," we conclude:

(A'). If $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, or $2p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$,

$$D = \prod_{\beta k} \sum_{s=1}^{\varphi(n)} \left(\prod_{k=1}^{m} \epsilon_k^{\,\mathrm{ind}_{g_k} a_s \cdot \beta_k} \right) L_{a_s},$$

the outer product extending over the ν sets of values of β_k for which $\sum_{k=1}^{m} \beta_k$ is odd.

(B'). If $n = 2^2 p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$,

$$D = \prod_{\beta k} \sum_{s=1}^{\varphi(n)} \left[(-1)^{(a_s-1)/2 \cdot \beta_0} \prod_{k=1}^{m} \epsilon_k^{\operatorname{ind}_{g_k} a_s \cdot \beta k} \right] L_{a_s},$$

the other product extending over the ν sets of values of βk for which $\sum_{k=0}^{m} \beta_k$ is odd.

(C'). If $n = 2^\lambda p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$, $\quad \lambda \geqq 3$,

$$D = \prod_{\beta k} \sum_{s=1}^{\varphi(n)} \left[(-1)^{(a_s-1)/2 \cdot \beta_{-1}} (\epsilon_0)^{\operatorname{ind}_3 a_s \cdot \beta_0} \prod_{k=1}^{m} \epsilon_k^{\operatorname{ind}_{g_k} a_s \cdot \beta k} \right] L_{a_s},$$

the outer product extending over the ν sets of values of β_k ($k = -1, 0, 1, 2, \cdots, m$) for which $\beta_{-1} + \sum_{i=1}^{m} \beta_i$ is odd, and $a_s \equiv \pm 5^{\operatorname{ind}_3 a_s}$, mod 2^λ.

We remark that if we had equated the exponents of $\mathfrak{q}_{a_k}$ in (3.3) for any ν ideals, say $\mathfrak{q}_{\alpha_i}$, $i = 1, 2, \cdots, \nu$, where $\alpha_1, \alpha_2, \cdots, \alpha_\nu$ represent ν of the numbers $a_1, a_2, \cdots, a_{\varphi(n)}$ chosen so that $\alpha_i \not\equiv -\alpha_j$, mod n ($i \neq j$), we would have obtained the equations:

$$\sum_{j=1}^{\nu} l_j (d_{\alpha_k \alpha_j^{-1}} - d_{\alpha_1 \alpha_j^{-1}}) = 0, \qquad k = 2, 3, \cdots, \nu.$$

D would become:

$$D' = | d_{\alpha_i \alpha_j^{-1}} |, \qquad i, j = 1, 2, \cdots, \nu,$$

which would have the same numerical value as before since the numbers α_i, $i = 1, 2, \cdots, \nu$, are congruent to $\pm a_k$, mod n, $k = 1, 2, \cdots, \nu$ in some order, so that, disregarding the order of the rows, the only difference between D and D' is that the signs of all the elements in some of the rows might be changed by virtue of: $d_{-a_s} = -d_{a_s}$. Since the exponents m_{a_k}, n_{a_k}, and S_{a_k} possess the characteristic properties of L_{a_k} we may state the following

LEMMA: *If a prime $q \equiv 1$, mod n, then in whatever way $q^{D/2^{\nu-1}}$ be represented as the product of two conjugate imaginary ideal factors in $k(e^{2\pi i/n})$ that are relatively prime, these factors are principal ideals, D being given by the expressions in A', B', C', above in which L_{a_k} is replaced by either m_{a_k}, n_{a_k}, or S_{a_k} defined in connection with* (1), (1.1), *and* (1.2), *respectively.*

7. The Case $n = p^k$, p an Odd Prime[20]

In the light of the above lemma, we first consider in this case the quantities n_{a_k}, taking $K = 0$ and $a \equiv 1$, mod n.[21] Then:

$$D/2^{r-1} = \prod_{\beta \text{odd}} \sum_{s=1}^{\varphi(n)} \epsilon^{\text{ind}_g a_s \cdot \beta} \mid a_s \mid /2^{r-1} = n^{r-1} \cdot H = p^{\mu} \cdot H,$$

where $\mu = k/2[p^{k-1}(p-1) - 2]$, and H is the "first-factor" of the class number for $k(e^{2\pi i/n})$. Now we consider the quantities S_{a_k}, taking $r = 0$ and $(M, n) = 1$. We have:

$$D/2^{r-1} = \prod_{\beta \text{odd}} \sum_{s=1}^{\varphi(n)} \epsilon^{\text{ind}_g a_s \cdot \beta} (M \mid a_s \mid - \mid M a_s \mid)/n \cdot (2n)^{r-1} =$$

$$\left(\prod_{\beta \text{odd}} (M - \epsilon^{-\text{ind}_g M \cdot \beta}) \Big/ n\right) H.$$

If now (ind M, $\varphi(n)$) $= d$, the numbers ind $M \cdot \beta$, β odd, fall into $\varphi(n)/d$ incongruent classes, mod $\varphi(n)$, consisting of $d/2$ numbers each, or $\varphi(n)/2d$ incongruent classes consisting of d numbers each according as $\varphi(n)/d$ is odd or even. Hence:

$$D/2^{r-1} = ((M^{\varphi(n)/d} - 1)^{d/2}/n)\, H, \text{ or, } ((M^{\varphi(n)2d} + 1)^d/n)H$$

according as $\varphi(n)/d$ is odd or even, M appertaining to the exponent $\varphi(n)/d$, mod n.[22]

We now choose $M = G$, where G is a primitive root of p for which $p^2 \nmid (G^{p-1} - 1)$.[23]

Then the factor multiplying H above becomes:

$$(G^{(p-1)/2 \cdot p^{k-1}} + 1)/p^k.$$

But: $G^{(p-1)/2} = -1 + Lp$, $p \nmid L$, whence our factor becomes:

$$((-1 + Lp)^{p^{k-1}} + 1)/p^k.$$

But:

$$(-1 + Lp^{\gamma})^p = -1 + mp^{\gamma+1}, \qquad p \nmid m, \gamma \geqq 1;$$

so that, by successive application:

$$(-1 + Lp^{\gamma})^{p^{\alpha}} = -1 + Np^{\gamma+\alpha}, \qquad p \nmid N.$$

Whence:

$$((-1 + Lp)^{p^{k-1}} + 1)/p^k = (-1 + Np^k + 1)/p^k = N.$$

Since $(N, p^{\mu}) = 1$, we have the

THEOREM: *If a prime $q \equiv 1$, mod p^k, p an odd prime, then, in the cyclotomic realm generated by $\omega = e^{2\pi i/p^k}$, no matter how q^H be represented as the product of*

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[20] To be consistent with my former notation I should write $p_1^{k_1}$. However, I have thought it best to omit the subscripts.

[21] Taking $(a, n) = 1$ would do just as well.

[22] By letting $M = n - 1$ we obtain $n^{r-1} \cdot H$ as above.

[23] Compare H. Weber, *loc. cit.*, V. 1, p. 686. The form of M justifies our choice.

two conjugate imaginary ideal factors that are relatively prime, these factors are principal ideals, H being the "first-factor" of the class number of the realm.

8. The Case $n = 2^\lambda$, $\lambda \geqq 3$

Referring again to the Lemma of Section 6, we have, in this case:

$$D/2^{\nu-1} = \prod_{\beta_0} \sum_{k=1}^{\varphi(n)} (-1)^{a_k - 1/2} (\epsilon_0)^{\operatorname{ind}_5 a_k \cdot \beta_0} S_{a_k}/2^{\nu-1} =$$

$$\left[\prod_{\beta_0} \sum_{k=1}^{\varphi(n)} (-1)^{a_k - 1/2} (\epsilon_0)^{\operatorname{ind}_5 a_k \cdot \beta_0} (M \mid a_k \mid - \mid M a_k \mid) \right] \Big/ n(2n)^{\nu-1},$$

letting $r = 0$. Let us choose M of the form $4m \pm 1$; then we have:

$$D/2^{\nu-1} = \left(\prod_{\beta_0} (M \mp \epsilon_0^{-\operatorname{ind}_5 M \cdot \beta_0})/2^{\lambda-1} \right) H.$$

If now $(\operatorname{ind}_5 M, 2^{\lambda-2}) = d$, the numbers $\operatorname{ind}_5 M \cdot \beta_0$ fall into $2^{\lambda-2}/d$ incongruent classes, mod $2^{\lambda-2}$, consisting of d numbers each, so that:

$$D/2^{\nu-1} = ([(\pm M)^{2^{\lambda-2}/d} - 1]^d/2^{\lambda-1}) H.$$

Choosing M as $2^\lambda - 1$, we have $\operatorname{ind}_5 M = 0$, whence $d = 2^{\lambda-2}$, and, therefore:

$$D/2^{\nu-1} = ((-M - 1)^{2^{\lambda-2}}/2^{\lambda-1}) H = 2^\mu H,$$

where $\mu = \lambda(2^{\lambda-2} - 1) + 1$.[24] Since the class number in this case is odd (H. Weber, foot-note 15), we may state the

THEOREM: *If a prime $q \equiv 1$, mod 2^λ, then, in the cyclotomic realm generated by a primitive (2^λ)th root of unity, no matter how q^n be represented as the product of two conjugate imaginary ideal factors that are relatively prime, these factors are principal ideals, H being the "first-factor" of the class number of the realm.*

Letting $M = 5$, above, we have:

$$D/2^{\nu-1} = (5^{2^{\lambda-2}} - 1/2^{\lambda-1}) H = 2K \cdot H,$$

where K is odd.[25] We will now show how this result may be obtained using the exponents m_{a_k}. We have:

$$D/2^{\nu-1} = \prod_{\beta_0} \sum_{k=1}^{\varphi(n)} (-1)^{(a_k-1)/2} \epsilon_0^{\operatorname{ind}_5 a_k \cdot \beta_0} (\mid a a_k \mid + \mid b a_k \mid - \mid (a+b) a_k \mid)/$$

$$n \cdot (2n)^{\nu-1}.$$

Let us choose: $a = 4s_1 + 1$, $b = 4s_2 + 1$, so that $s_1 + s_2$ is odd. Then $a + b = 2N$, where N is of the form $4s_3 - 1$. The numerator on the right-hand side above then becomes:

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[24] This result can be obtained by using the exponents n_{ak} as in the last section.

[25] See, for example, H. Weber, *loc. cit.*, V. 2, p. 65.

$$\prod_{\beta_0}\Bigg[(\epsilon_0^{-\mathrm{ind}_5 a\cdot\beta_0}+\epsilon_0^{-\mathrm{ind}_5 b\cdot\beta_0})\sum_{k=1}^{\varphi(n)}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}a_k$$
$$+\epsilon_0^{-\mathrm{ind}_5 N\cdot\beta_0}\sum_{k=1}^{\varphi(n)}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}\,|\,2a_k\,|\Bigg],$$

$N\cdot a_k$ being of the form $4m_1\pm 1$ when a_k is of the form $4m_2\mp 1$. But:

$$\epsilon_0^{-\mathrm{ind}_5 N\cdot\beta_0}\sum_{k=1}^{\varphi(n)}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}\,|\,2a_k\,|=$$
$$\epsilon_0^{-\mathrm{ind}_5 N\cdot\beta_0}\Bigg[2\sum_{k=1}^{\nu}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}\cdot a_k$$
$$+2\sum_{k=\nu+1}^{\varphi(n)}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}(a_k-2^{\lambda-1})\Bigg].$$

Now $a_k-2^{\lambda-1}\equiv a_k(1-2^{\lambda-1})$, mod 2^λ, since a_k is odd, whence:

$$\mathrm{ind}_5(a_k-2^{\lambda-1})=\mathrm{ind}_5 a_k+\mathrm{ind}_5(2^{\lambda-1}-1)=\mathrm{ind}_5 a_k+2^{\lambda-3}.$$

Therefore, for β_0 odd, we have:

$$2\sum_{k=\nu+1}^{\varphi(n)}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}(a_k-2^{\lambda-1})=-2\sum_{k=1}^{\nu}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}\cdot a_k.$$

For β_0 even, we have:

$$2\sum_{k=\nu+1}^{\varphi(n)}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}(a_k-2^{\lambda-1})=$$
$$2\sum_{k=\nu+1}^{\varphi(n)}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}\cdot a_k+2^\lambda\sum_{k=1}^{\nu}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0},$$

by virtue of:

$$\sum_{k=1}^{\varphi(n)}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}=0.$$

Now:

$$\sum_{k=1}^{\nu}(-1)^{(a_k-1)/2}\epsilon_0^{\mathrm{ind}_5 a_k\cdot\beta_0}=\sum_{\text{odd } s=1}^{2^{\lambda-1}-1}(-1)^{(s-1)/2}\epsilon_0^{\mathrm{ind}_5 s\cdot\beta_0}=$$
$$\sum_{\text{odd } t=1}^{2^{\lambda-2}-1}(-1)^{t-1/2}[\epsilon_0^{\mathrm{ind}_5 t\cdot\beta_0}-\epsilon_0^{\mathrm{ind}_5(2^{\lambda-1}-t)\cdot\beta_0}]=$$
$$\sum_{\text{odd } t=1}^{2^{\lambda-2}-1}(-1)^{(t-1)/2}(\epsilon_0^{\mathrm{ind}_5 t\cdot\beta_0}-\epsilon_0^{\mathrm{ind}_5 t\cdot\beta_0})=0,$$

β_0 being even. We therefore conclude:

$$D/2^{r-1} = \Big(\prod_{\beta_0 \text{ odd}} (\epsilon_0^{-\text{ind}_s a\cdot\beta_0} + \epsilon_0^{-\text{ind}_s b\cdot\beta_0}) \cdot$$

$$\prod_{\beta_0 \text{ even}} (\epsilon_0^{-\text{ind}_s a\cdot\beta_0} + \epsilon_0^{-\text{ind}_s b\cdot\beta_0} + 2\epsilon_0^{-\text{ind}_s N\cdot\beta_0})/2^{\lambda-1}\Big) H,$$

a and b being chosen as above and $N = (a + b)/2$.

We now let $a \equiv 1, b \equiv 2^\lambda - 3, \text{mod } 2^\lambda$, so that:

$$N = (a + b)/2 \equiv 2^{\lambda-1} - 1, \text{mod } 2^\lambda.$$

The factor multiplying H above becomes:

$$\prod_{\beta_0 \text{ odd}} (1 + \epsilon_0^{-\text{ind}_s 3\cdot\beta_0}) \prod_{\beta_0 \text{ even}} (3 + \epsilon_0^{-\text{ind}_s 3\cdot\beta_0})/2^{\lambda-1}.$$

Taking $\lambda > 3$, and observing that $\text{ind}_5 3$ is odd, our factor becomes:

$$\prod_{\beta_0 \text{ odd}} (-1 - \epsilon^{\beta_0}) \prod_{\beta_0 \text{ even}} (-3 - \epsilon^{\beta_0})/2^{\lambda-1} =$$

$$[(-1)^{2^{\lambda-3}} + 1]\,[(-3)^{2^{\lambda-3}} - 1]/2^{\lambda-1} = [2]\,[(3)^{2^{\lambda-3}} - 1]/2^{\lambda-1},$$

ϵ being a primitive root of $x^{2^{\lambda-2}} - 1 = 0$. Since:

$$3^{2^{\lambda-3}} - 1 = K' \cdot 2^{\lambda-1}$$

where K' is odd, the factor reduces to $2K'$.

9. Two Consequences and an Upper Bound for the Numerical Value of the "First-Factor" of the Class Number

From Section 2 and the subsequent development in Section 5, we may consider the principal ideals mentioned in the Lemma of Section 6 as being generated by a product of powers of Ψ-functions. This remark would also hold true for the principal ideals mentioned in the theorems of Sections 7 and 8, remembering that, if a and b are relatively prime, there exist two integers x and y such that $ax + by = 1$. If then, in the cyclotomic realm $k(\alpha = e^{2\pi i/p^s})$, p an odd prime,

$$[q^H] = [F(\alpha)]\,[F(\alpha^{-1})],$$

where q is a prime $\equiv 1, \text{mod } p^s$, and $F(\alpha)$ is a product of powers of Ψ-functions, we must have:

$$q^H = F(\alpha)F(\alpha^{-1}),$$

since $\Psi(\alpha)\Psi(\alpha^{-1}) = q$. (Section 2, property (a).) Any other factorization of q^H into two conjugate imaginary factors that are relatively prime, in which one of the factors is associated with $F(\alpha)$ the other with $F(\alpha^{-1})$, must be of the form

$$\epsilon(\alpha)F(\alpha)\epsilon(\alpha^{-1})F(\alpha^{-1}),$$

$\epsilon(\alpha)$ being a unit. Whence:

$$\epsilon(\alpha)\epsilon(\alpha^{-1}) = 1,$$

and $\epsilon(\alpha)$ is a root of unity.[26] But the only roots of unity in $k(\alpha)$ are $\pm \alpha^l$, $0 \leqq l < p^s$.[27] Since there are $2^{\nu-1}$ ways of representing $[q^H]$ as the product of two conjugate imaginary ideal factors that are relatively prime ($2\nu = \varphi(p^s)$), we have the

THEOREM: *If a prime $q \equiv 1$, mod p^s, then, in the cyclotomic realm $k(\alpha = e^{2\pi i/p^s})$, q^H will admit $2^{\nu-1} \cdot p^s$ factorizations into two conjugate imaginary factors that are relatively prime, allowing that two factors differing merely in sign are the same, H being the "first-factor" of the class number of the realm.*

Since the only roots of unity in $k(\alpha = e^{2\pi i/2^\lambda})$ are α^l $(0 \leqq l < 2^\lambda)$, a similar theorem holds in this realm.

Considering now $k(\zeta = e^{2\pi i/p^s})$, p any prime, suppose

$$\left[\prod_{i=1}^{\nu} \mathfrak{q}_{\alpha_i}\right]^H = [F(\zeta)],$$

the $\mathfrak{q}$'s being defined as heretofore, and $\alpha_1, \alpha_2, \cdots, \alpha_\nu$ are ν numbers chosen from the $\varphi(p^s)$ totitives so that $\alpha_i \not\equiv -\alpha_j$, mod $p^s (i \neq j)$. If we replace one of the ideals $\mathfrak{q}_{\alpha_i}$ on the left by $\mathfrak{q}_{-\alpha_i}$ and form the quotient, we have:

$$\mathfrak{q}_{\alpha_i}^H / \mathfrak{q}_{-\alpha_i}^H = [F(\zeta)]/[F'(\zeta)],$$

and therefore we may state the

THEOREM: *If a prime $q \equiv 1$, mod p^s, and, in $k(\alpha = e^{2\pi i/p^s})$,*

$$[q] = \prod_{i=1}^{\varphi(p^s)} \mathfrak{q}_{\alpha_i},$$

then $\mathfrak{q}_{\alpha_i}^H$ and $\mathfrak{q}_{-\alpha_i}^H$ belong to the same ideal class, $i = 1, 2, \cdots, \varphi(p^s)$.

In the light of the Lemma of Section 6, by using the exponents m_{a_k} we can find an upper bound for $|H|$ in case $n = p^k$, p any prime. Thus, in this case, we have:

$$d_{a_k} = m_{a_k} - m_{-a_k} = \pm 1$$

according as $|aa_k| + |ba_k| \gtrless n$. Hence:

$$|D| = ||d_{a_i a_j^{-1}}|| \leqq \nu!, \qquad i, j = 1, 2, \cdots, \nu.$$

Now, for $n = p^k$, p an odd prime, using the exponents m_{a_k} we obtain the result:

$$D/2^{\nu-1} = \left(\prod_{\beta \text{ odd}} [\epsilon^{-\text{ind}_g a \cdot \beta} + \epsilon^{-\text{ind}_g b \cdot \beta} - \epsilon^{-\text{ind}_g (a+b) \cdot \beta}]/n\right) H,$$

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[26] D. Hilbert, *loc. cit.*, Theorem 48, Article 21. Compare also L. Kronecker, *Oeuvres*, V. 1, p. 103, 1857. H. Minkowski, *Geometrie der Zahlen*. Leipzig, 1896.

[27] R. Fricke, *loc. cit.*, p. 200.

after choosing a, b, $(a + b)$ prime to p. By making a certain choice for a, b we can obtain an upper bound for $| H |$. For example, choosing $a \equiv b \equiv 1$, mod n, we obtain:

$$D/2^{\nu-1} = ((2^{\varphi(n)/d} - 1)^{d/2}/n)H, \text{ or, } ((2^{\varphi(n)/2d} + 1)^d/n)H$$

according as $\varphi(n)/d$ is odd or even, where $(\text{ind}_g 2, \varphi(n)) = d$. Therefore:

$$| H | \leqq n \cdot \nu!/2^{\nu-1}(2^{\varphi(n)/d} - 1)^{d/2}, \text{ or, } \leqq n \cdot \nu!/2^{\nu-1}(2^{\varphi(n)/2d} + 1)^d$$

according as $\varphi(n)/d$ is odd or even. For $n = 2^\lambda$, $\lambda > 3$, we can, for example, make use of a result of Section 8, viz:

$$D/2^{\nu-1} = ([2]\ [(3)^{2^{\lambda-3}} - 1]/2^{\lambda-1})H.$$

We then have:

$$| H | \leqq \nu!/2^{\nu-\lambda+1}[(3)^{2^{\lambda-3}} - 1].$$

10. **A Remark Involving the Exponents m_{a_k} When $(a, n) = (b, n) = r_i$** [28]

From relations (3.2) replacing L_{a_k} by m_{a_k}, we have:

$$\left(\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{m_{a_k} a_j^{-1}}\right)^{l_j} \sim 0,$$

where $m_{a_k} = (| aa_k | + | ba_k | - | (a + b)a_k |)/n$. Suppose, now, $n = r_i s_i$, $(r_i, s_i) = 1$. Further, let a and b be chosen so that: $(a, n) = (b, n) = r_i$. Let $a_k \equiv \alpha_k$, mod s_i, where α_k is the least positive residue of a_k, mod s_i. If, now, $a_k \equiv a_j$, mod $s_i (k \neq j)$, then:

$$m_{a_k} = m_{a_j} = (| a\alpha_k | + | b\alpha_k | - | (a + b)\alpha_k |)/n = \mu_{a_k},$$

say. Suppose now that the ν numbers $a_1, a_2, \cdots, a_\nu$ have the s distinct numbers $\beta_1, \beta_2, \cdots, \beta_s$ among α_i, $i = 1, 2, \cdots, \nu$, as their least positive residues, mod s_i, where $a_1 = \alpha_1 = \beta_1 = 1$. Let $\beta_j \equiv a_j$, mod s_i, so that, $\beta_j^{-1} \equiv a_j^{-1}$, mod s_i, the Greek letters denoting least positive residues, mod s_i, and, we have:

$$\left(\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{m_{a_k} a_j^{-1}}\right)^{l_j} = \left(\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{\mu_{a_k} \beta_j^{-1}}\right)^{l_j} \sim 0.$$

Forming the product for $j = 1, 2, \cdots, s$, we obtain:

$$\prod_{k=1}^{\varphi(n)} \mathfrak{q}_{a_k}^{\sum_{j=1}^{s} \mu_{a_k} \beta_j^{-1} \cdot l_j} \sim 0.$$

Equating the exponents for $k = 1, 2, \cdots, \nu$, we obtain the $(s - 1)$ equations:

$$\sum_{j=1}^{s} l_j(\mu_{\beta_k \beta_j^{-1}} - \mu_{\beta_1 \beta_j^{-1}}) = 0, \qquad k = 2, 3, \cdots, s.$$

[28] A similar remark applies to the exponents n_{a_k}.

Equating them for $k = \nu + 1, \cdots, \varphi(n)$, we obtain the $(s-1)$ equations:

$$\sum_{j=1}^{s} l_j(\mu_{-\beta_k\beta_j^{-1}} - \mu_{-\beta_1\beta_j^{-1}}) = 0, \qquad k = 2, 3, \cdots, s.$$

Setting $\mu_x - \mu_{-x} = \delta_x$, and, remembering that $\mu_x + \mu_{-x} = 1$, we find that both sets of equations above are equivalent to:

$$(5') \qquad \sum_{j=1}^{s} l_j(\delta_{\beta_k\beta_j^{-1}} - \delta_{\beta_1\beta_j^{-1}}) = 0, \qquad k = 2, 3, \cdots, s.$$

The power to which $\prod_{k=1}^{\nu} q_{a_k}$ must be raised to give the principal class is now:

$$\sum_{j=1}^{s} l_j\delta_{\beta_j^{-1}}.$$

Equations (5') always have common solutions other than 0. If they are linearly independent the solutions l_j are proportional to the co-factors of $\delta_{\beta_j^{-1}}$ in the first row of

$$D' = \begin{vmatrix} \delta_{\beta_1\beta_1^{-1}} & \delta_{\beta_1\beta_2^{-1}} \cdots\cdots\cdots & \delta_{\beta_1\beta_s^{-1}} \\ \delta_{\beta_2\beta_1^{-1}} - \delta_{\beta_1\beta_1^{-1}} & \cdots\cdots\cdots & \delta_{\beta_2\beta_s^{-1}} - \delta_{\beta_1\beta_s^{-1}} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \delta_{\beta_s\beta_1^{-1}} - \delta_{\beta_1\beta_1^{-1}} & \cdots\cdots\cdots & \delta_{\beta_s\beta_s^{-1}} - \delta_{\beta_1\beta_s^{-1}} \end{vmatrix}.$$

Therefore:

$$\sum_{j=1}^{s} l_j\delta_{\beta_j^{-1}} = K \cdot D'.$$

Now

$$D' = |\, \delta_{\beta_i\beta_j^{-1}} \,|, \qquad i, j = 1, 2, \cdots, s,$$

so that, if among the numbers $\beta_1, \beta_2, \cdots, \beta_s$ we have: $\beta_l \equiv -\beta_k$, mod $s_i (l \neq k)$, then $D' = 0$, since $\delta_{-x} = -\delta_x$. We therefore conclude that a necessary condition that

$$\sum_{j=1}^{s} l_j\delta_{\beta_j^{-1}} \not\equiv 0$$

is that either equations (5') be linearly dependent, or, among the numbers $\beta_1, \beta_2, \cdots, \beta_s$, there exist no congruence:

$$\beta_l \equiv -\beta_k, \text{ mod } s_i (l \neq k).$$

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ON THE NEVANLINNA AND BOSANQUET-LINFOOT SUMMATION METHODS

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1. **Introduction.** In part I of this paper we give theorems concerning the inclusion of the Bosanquet-Linfoot summation method (the $B_{\alpha,\beta}$ method)[1] in our Nevanlinna summation method (the N_{q_p} method).[2] We show, incidentally, that the specialized N_{q_0} method which Nevanlinna states (without proof) to be weaker than the Cesàro method (C, α), $\alpha > 0$,[3] is, as a consequence of properties of the $B_{\alpha,\beta}$ method, weaker than any (C, α), $\alpha > 0$.

In part II we modify somewhat the defining conditions of an N_{q_p} kernel and thus obtain a summation method (the N_{z_p} method) which is more general and inclusive than the N_{q_p} method. We give, also, theorems concerning the effectiveness of the N_{z_p} method for the summation of Fourier series.

I. On Inclusion of the $B_{\alpha,\beta}$ Method by the N_{q_p} Method

2. **The N_{q_p} and $B_{\alpha,\beta}$ methods.** For the convenience of the reader we define here the $B_{\alpha,\beta}$ method and give some of the properties of the N_{q_p} and $B_{\alpha,\beta}$ methods which we later refer to.

Let $F(n)$ be a function of n defined for all values of $n > 0$ and let it be desired to consider a "generalized limit" of $F(n)$ as $n \to \infty$. The function $F(n)$ may represent the n^{th} partial sums of a series which have been defined for non-integral values of n by some method such as linear interpolation.

The $B_{\alpha,\beta}$ method is given by

DEFINITION 2.1. *The $B_{\alpha,\beta}$ limit of $F(n)$ as $n \to \infty$ is said to be S if, for every sufficiently large C,*

$$\lim_{n\to\infty} B_{\alpha,\beta}\, F(n) \equiv \lim_{n\to\infty} \int_0^1 B_{\alpha,\beta}\,(t)\, F(nt)\, dt = S,$$

where

$$B_{\alpha,\beta}(t) = \begin{cases} G(1-t)^{\alpha-1} (\log C/1-t)^{-\beta} & (\alpha > 0) \\ G(1-t)^{-1} (\log C/1-t)^{-\beta-1} & (\alpha = 0, \beta > 0), \end{cases}$$

[1] L. S. Bosanquet and E. H. Linfoot, Generalized Means and the Summability of Fourier Series, Quarterly Journal of Mathematics (Oxford Series), Vol. 2 (1931), pp. 207–229.

[2] A. F. Moursund, On a Method of Summation of Fourier Series (Second Paper), Annals of Mathematics (2), Vol. 34 (1933), pp. 778–798.

[3] F. Nevanlinna, Über die Summation der Fourier'schen Reihen und Integrale, Översikt av Finska Vetenskaps-Societetens-Förhandlingar, 64, A, No. 3, 14 pp.

the function $G = G(\alpha, \beta, C)$ *being so chosen that*

$$\int_0^1 B_{\alpha,\beta}(t)\, dt = 1.^4$$

It follows from the definition of $G(\alpha, \beta, C)$ that

$$G(0, \beta, C) = \beta(\log C)^\beta. \tag{2.1}$$

The following lemmas give general properties of the $B_{\alpha,\beta}$ method.[5]

LEMMA 2.1. *To each* $B_{\alpha,\beta}(t)$ *with* $\alpha > 0$ *or* $\alpha = 0$, $\beta > 0$ *corresponds a* C_0 $(\alpha, \beta) \geqq 1$ *such that the existence of* $\lim B_{\alpha,\beta} F(n)$ *as* $n \to \infty$ *for some* $C > C_0$ *implies its existence for every* $C > 1$.

It follows from Lemma 2.1 that in Definition 2.1 the phrase "for every sufficiently large C" may be replaced by either of the phrases "for an arbitrarily large C" or "for every $C > 1$."

LEMMA 2.2. *If* $B_{\alpha,\beta} F(n) \to S$ *as* $n \to \infty$, *where either* $\alpha > 0$, *or* $\alpha = 0$, $\beta \geqq 0$, *then* $B_{\alpha',\beta'} F(n) \to S$ *as* $n \to \infty$ *provided either* $\alpha' > \alpha$ *or* $\alpha' = \alpha, \beta' > \beta$.

Since the N_{q_p} method, $p = 0, 1, 2, \cdots$, is defined in these Annals,[6] we do not repeat the definition here. The essential difference between the N_{q_p} and $B_{\alpha,\beta}$ methods lies in the fact that the kernel $q_p(t)$ of the N_{q_p} method permits of further specialization, while the kernel $B_{\alpha,\beta}(t)$ of the $B_{\alpha,\beta}$ method is a completely defined function.

Nevanlinna states without proof that the choice of the function

$$q_0^*(t) \equiv (1 + \epsilon)(\log a)^{1+\epsilon}(1 - t)^{-1}(\log a/1 - t)^{-2-\epsilon},$$

with $\epsilon > 0$ and $a \geqq e^{2+\epsilon}$, as kernel for the N_{q_0} method gives a method of summation weaker than any (C, α), $\alpha > 0$.[7] Setting $1 + \epsilon = \beta$ and $a = C$, we have

$$q_0^*(t) \equiv \beta(\log C)^\beta (1 - t)^{-1}(\log C/1 - t)^{-\beta-1}. \tag{2.2}$$

3. **Preliminary lemmas.** We will use the lemmas of this section in our proofs of the theorems of Part I.

LEMMA 3.1. *For* $C \geqq e$ *and* $i \geqq 1$,

$$\frac{d^i}{dt^i}(\log C/1 - t)^{-\beta}$$

$$= -\beta(i - 1)!\,(1 - t)^{-i}(\log C/1 - t)^{-\beta-1}\{1 + O[(\log C/1 - t)^{-1}]\}.$$

---

[4] Bosanquet and Linfoot, *loc. cit.*[1] set up two equivalent summation methods. We define here and use throughout the paper their second method.

[5] Bosanquet and Linfoot, *loc. cit.*[1] In proving the regularity part of Lemma 2.2 they assume that $F(n)$ is bounded.

[6] Moursund, *loc. cit.*[2], § 2.

[7] *Loc. cit.*[3] This kernel gives a regular, but not an N_{q_0}, method of summation when $-1 < \epsilon \leqq 0$. For details of and theorems concerning the N_{q_0} method see our paper, On a Method of Summation of Fourier Series, These Annals, Vol. 33 (1932), pp. 773–784. The method is there called the N_q method.

Proof. Since, for $j \geqq 1$,

$$(1-t)^j \frac{d^j}{dt^j} (\log C/1-t)^m = O[(\log C/1-t)^{m-1}],$$

we have, using Leibniz's formula for the n^{th} derivative of a product,

$$\frac{d^i}{dt^i}\left(\log \frac{C}{1-t}\right)^{-\beta} = \frac{d^{i-1}}{dt^{i-1}}\left[-\beta(1-t)^{-1}\left(\log \frac{C}{1-t}\right)^{-\beta-1}\right]$$

$$= -\beta(i-1)!(1-t)^{-i}\left\{\left(\log \frac{C}{1-t}\right)^{-\beta-1} + \sum_{j=1}^{i-1} \frac{(1-t)^j}{j!} \frac{d^j}{dt^j}\left(\log \frac{C}{1-t}\right)^{-\beta-1}\right\}$$

$$= -\beta(i-1)!(1-t)^{-i}\left(\log \frac{C}{1-t}\right)^{-\beta-1}\left\{1 + O\left[\left(\log \frac{C}{1-t}\right)^{-1}\right]\right\}.$$

Making use of Lemma 3.1, we see that

LEMMA 3.2. *For $C \geqq e$, $\alpha > 0$, and $r \geqq 1$,*

$$\frac{d^r}{dt^r} B_{\alpha,\beta}(t) = G(1-t)^{\alpha-1-r}\left(\log \frac{C}{1-t}\right)^{-\beta}$$

$$\left\{(-1)^r(\alpha-1)(\alpha-2)\cdots(\alpha-r)\right.$$

$$+(-\beta)(-1)^r\left(\log \frac{C}{1-t}\right)^{-1} \sum_{i=1}^{r} \frac{(-1)^r r!}{i(r-i)!}(\alpha-1)(\alpha-2)\cdots(\alpha-r+i)$$

$$\left. + O\left[\left(\log \frac{C}{1-t}\right)^{-2}\right]\right\},$$

where when $i = r$ the factor $(\alpha-1)(\alpha-2)\cdots(\alpha-r+i)$ must be replaced by 1.

Then:

LEMMA 3.3. *If $p \geqq 1$ and $0 \leqq r < p$, $\frac{d^r}{dt^r} B_{\alpha,\beta}(t) = 0$ when $t = 1$, provided C is sufficiently large and either $\alpha = p, \beta > 0$ or $\alpha > p$.*

LEMMA 3.4. *If $p \geqq 0$ and $0 < t < 1$, then $(-1)^p \frac{d^p}{dt^p} B_{\alpha,\beta}(t) \geqq 0$, provided C is sufficiently large and either $\alpha = p, \beta > 0$ or $\alpha > p$.*

Proof. We see that the lemma holds for $p = 0$ by inspection of $B_{\alpha,\beta}(t)$ [Definition 2.1]. For $p \geqq 1$, $(-1)^p \frac{d^p}{dt^p} B_{\alpha,\beta}(t)$ is non-negative for we see using Lemma 3.2 that when $\alpha = p, \beta > 0$ its sign is determined by

$$(-1)^p\left\{-\beta(-1)^p \sum_{i=1}^{p} \frac{(-1)^i p!}{i(p-i)!} \frac{(p-1)!}{(i-1)!}\right\} = \beta(p-1)! > 0:$$

and, when $\alpha > p$, by

$$(-1)^p \{(-1)^p (\alpha - 1)(\alpha - 2) \cdots (\alpha - p)\} > 0.$$

LEMMA 3.5. *If* $p \geqq 0$ *and* $0 < t < 1$, *then* $(-1)^p \frac{d^{p+1}}{dt^{p+1}} B_{\alpha,\beta}(t) \geqq 0$, *provided C is sufficiently large and*

Case 1. $\alpha = p, \beta > 1$;
Case 2. $p < \alpha < p + 1$;
Case 3. $\alpha = p + 1, \beta \leqq 0$.

Proof. When $\alpha = 0$ we have

$$\frac{d}{dt} B_{0,\beta}(t) = G(1-t)^{-2}(\log C/1-t)^{-\beta-1}[1 - (\beta+1)(\log C/1-t)^{-1}] > 0.$$

When $\alpha > 0$, $(-1)^p \frac{d^{p+1}}{dt^{p+1}} B_{\alpha,\beta}(t)$ is non-negative, for we see using Lemma 3.2 that its sign is determined by:

Case 1. $$(-1)^p \left\{ -\beta(-1)^{p+1} \sum_{i=2}^{p+1} \frac{(-1)^i (p+1)!}{i(p+1-i)!} \frac{(p-1)!}{(i-2)!} \right\}$$

$$= \beta(p-1)! \sum_{i=2}^{p+1} \frac{(-1)^i (p+1)!}{i!(p+1-i)!} (i-1) = \beta(p-1)! > 0;$$

Case 2. $(-1)^p \{(-1)^{p+1}(\alpha - 1)(\alpha - 2) \cdots (\alpha - p)(\alpha - p - 1)\} > 0$, the factor $(\alpha - p - 1)$ being negative;

Case 3. $$(-1)^p \left\{ -\beta(-1)^{p+1} \sum_{i=1}^{p+1} \frac{(-1)^i (p+1)!}{i(p+i-i)!} \frac{p!}{(i-1)!} \right\} = -\beta p! \geqq 0.$$

LEMMA 3.6 *If* $p \geqq 0$ *and* $0 < t < 1$, *then* $(-1)^p \frac{d^{p+1}}{dt^{p+1}} B_{\alpha,\beta}(t) \leqq 0$, *provided C is sufficiently large and* $\alpha = p + 1, \beta > 0$.

Proof. Follows from inspection of the proof of Lemma 3.4 Case 3.

LEMMA 3.7. *The limit*

$$\lim_{\delta \to 0} \int_0^{1-\delta} \frac{d^p}{dt^p} B_{\alpha,\beta}(t) \log 1/1 - t \, dt$$

exists for Cases 1, 2, *and* 3 *of Lemma* 3.5.

Proof. It follows from the definition of $B_{\alpha,\beta}(t)$ that the limit exists when $p = 0$. The limit also exists when $p > 0$ for we see using Lemma 3.2 that its existence depends upon the existence of:

Case 1. $\lim_{\delta \to 0} \int_0^{1-\delta} (1-t)^{-1} (\log C/1-t)^{-\beta-1} \log 1/1 - t \, dt$, where $\beta > 1$;

Case 2. $\lim_{\delta \to 0} \int_0^{1-\delta} (1-t)^{\alpha-1-p} (\log C/1-t)^{-\beta} \log 1/1-t\, dt,$

where $a - 1 - p > -1$;

Case 3. $\lim_{\delta \to 0} \int_0^{1-\delta} (\log C/1-t)^{-\beta} \log 1/1-t\, dt$; all of which limits exist.

LEMMA 3.8. *The limit*

$$\lim_{\delta \to 0} \int_0^{1-\delta} \frac{d^{p+1}}{dt^{p+1}} B_{\alpha,\beta}(t) \log 1/1-t\, dt$$

does not exist when $\alpha = p + 1, \beta \leqq 1$.

Proof. We see using Lemma 3.2 that the limit depends upon the existence of

$$\lim_{\delta \to 0} \int_0^{1-\delta} (1-t)^{-1} (\log C/1-t)^{-\beta-1} \log 1/1-t\, dt,$$

which does not exist for $\beta \leqq 1$.

4. **Theorems concerning $B_{\alpha,\beta}$ and N_{qp} kernels.** The following theorems show when a $B_{\alpha,\beta}$ kernel is an N_{qp} kernel.[8]

THEOREM 4.1. *A $B_{\alpha,\beta}$ kernel is an N_{qp} kernel for each of the three cases.*

Case 1. $\alpha = p, \beta > 1$;
Case 2. $p < \alpha < p + 1$;
Case 3. $\alpha = p + 1, \beta \leqq 0$;
provided C is sufficiently large.

Proof. In all three cases it follows from the definition of $B_{\alpha,\beta}(t)$ and Lemmas 3.3, 3.4, 3.5, and 3.7 that, for every specified $p \geqq 0$, the function $B_{\alpha,\beta}(t)$ satisfies the conditions which define an N_{qp} kernel.[8]

THEOREM 4.2. *A $B_{0,\beta}$ kernel, $\beta \leqq 1$, is not an N_{q0} kernel; and a $B_{p+1,\beta}$ kernel, $0 < \beta \leqq 1$, is neither an N_{qp} kernel nor an N_{qp+1} kernel.*

Proof. The first part of the theorem follows from the fact that

$$\lim_{\delta \to 0} \int_0^{1-\delta} G(1-t)^{-1} (\log C/1-t)^{-\beta-1} \log 1/1-t\, dt$$

does not exist for $\beta \leqq 1$. We obtain the second part by using Lemmas 3.6 and 3.8 and referring to Conditions (6) and (7) in the definition of the N_{qp} kernel.[9]

THEOREM 4.3. *For $\epsilon > -1$ and $a \geqq e^{2+\epsilon}$, the function $q_0(t)$ [Nevanlinna's weak kernel] is a $B_{0,\beta}$ kernel, $\beta > 0$.*

Proof. Follows from the definition of $B_{0,\beta}(t)$, (2.1), and (2.2).

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[8] Moursund, *loc. cit.*[2], §2.
[9] Moursund, *loc. cit.*[2], §2.

5. **Final theorems of Part I.** As a consequence of Theorem 4.1 (we need really only Case I) and Lemma 2.2, we have

THEOREM 5.1. *Whenever the N_{q_p} limit of $F(n)$ as $n \to \infty$ exists, the $B_{\alpha,\beta}$ limit of $F(n)$ as $n \to \infty$ also exists, provided either $\alpha = p, \beta > 1$ or $\alpha > p$.*

The above theorem enables us to state

THEOREM 5.2. *By replacing N_q by $B_{\alpha,\beta}$, $\alpha = 0, \beta > 1$ or $\alpha > 0$ in the statements of Theorems* 6.1, 6.2, 6.3, 6.4, *and* 7.1 *of our first paper*[10] *and by replacing, for every specified p, N_{qp} by $B_{\alpha,\beta}$, $\alpha = p, \beta > 1$ or $\alpha > p$ in the statements of Theorems* 9.1, 9.2, 9.3, *and* 9.4 *of our second paper*,[11] *theorems are obtained concerning the effectiveness of the $B_{\alpha,\beta}$ method for summing Fourier series, conjugate Fourier series, and the derived series of such series.*

Among the theorems thus obtained for the $B_{\alpha,\beta}$ method are theorems equivalent to Theorems 3.1 and 3.2 given by Bosanquet and Linfoot for the $B_{0,\beta}$ method,[12] and theorems equivalent to all of the theorems of Smith.[13] Our proofs of these theorems are simpler than those given by Bosanquet and Linfoot and Smith, who use asymptotic expansions obtained by contour integration in their proofs.

THEOREM 5.3. *For $\epsilon > 0$ and $a \geqq e^{2+\epsilon}$, kernels of the type $q_0^*(t)$ give an N_{q_0} method of summation which is weaker than any (C, α) $\alpha > 0$.*[14]

Proof. Follows from Theorem 4.3 and Lemma 2.2.

II. On the N_{z_p} Method

6. **The N_{z_p} Method.** We define here the N_{z_p} method mentioned in the introduction, and give some of its general properties. The N_{z_p} method while essentially like the N_{q_p} method has the advantage of admitting a larger class of functions as kernel.

DEFINITION 6.1. *The N_{z_p} limit of $F(n)$ as $n \to \infty$ is said to be S if the limit*

$$\lim_{n\to\infty} N_{z_p}F(n) \equiv \lim_{n\to\infty} \int_0^1 z_p(t)\, F(nt)dt = S,$$

where, for $p = 1, 2, \cdots$,

$$z_p(t) \geqq 0 \text{ for } 0 \leqq t \leqq 1; \tag{1}$$

$$\int_0^1 z_p(t)dt = 1; \tag{2}$$

[10] *Loc. cit.*[7]

[11] *Loc. cit.*[2]

[12] L. S. Bosanquet and E. H. Linfoot, On the Zero Order Summability of Fourier Series, London Mathematical Society Journal, Vol. 6 (1931), pp. 117–126.

[13] A. H. Smith, On the Summability of Derived Series of the Fourier Lebesgue Type, Quarterly Journal of Mathematics (Oxford Series), Vol. 4 (1933), pp. 93–106.

[14] See footnote 7. Bosanquet and Linfoot, *loc. cit.*[12] give an example of a continuous function $f(x)$ whose Fourier series diverges $B_{0,1}$ at the point $x = 0$. It follows that when $-1 < \epsilon \leqq 0$, kernels of the type $q_0^*(t)$, being $B_{0,\beta}$ kernels, $0 < \beta \leqq 1$, give a method of summation weaker than any (C, α), $\alpha > 0$.

(3) $\frac{d^r}{dt^r} z_p(t)$ *exists and is continuous for* $0 \leqq t \leqq 1, \qquad (r = 1, 2, \cdots, p-1);$

(4) $\frac{d^r}{dt^r} z_p(t) = 0$ *for* $t = 1, \qquad (r = 0, 1, 2, \cdots, p-1);$

(5) $\frac{d^p}{dt^p} z_p(t)$ *exists for* $0 < t < 1$ *and is integrable over* $(0, 1);$

(6) $$\left| \int_0^1 t^p \frac{d^p}{dt^p} z_p(t) \begin{matrix} \sin \\ \cos \end{matrix} vt \, dt \right| \leqq Z_p(v) \text{ for } v \geqq V_0$$

(V_0 *some constant*), *where*

(a) $Z_p(v)$ *is an absolutely continuous monotone decreasing function of* v *for* $v \geqq V_0$,

(b) $$\int_{V_0}^{\infty} Z_p(v)/v \, dv \text{ exists.}$$

From assumptions (a) and (b) it follows that

(c) $$Z_p(v) \to 0 \text{ as } v \to \infty.$$

If $p = 0$, we assume that $z_0(t)$ is non-negative for $0 < t < 1$ and satisfies (2) and (6).

There is no loss of generality if it is assumed that in (6) the constant $V_0 = \pi$.

The following lemmas give properties of the N_{z_p} method.

LEMMA 6.1. *The* N_{z_p} *method is regular.*[15]

LEMMA 6.2. *Every* N_{z_p} *kernel is an* N_{z_r} *kernel for* $r = 0, 1, 2, \cdots, p-1$.

Proof. For $r < p$ we have

$$\left| \int_0^1 t^r \frac{d^r}{dt^r} z_p(t) \begin{matrix} \sin \\ \cos \end{matrix} vt \, dt \right| \leqq K/v.$$

The function K/v may be taken as $Z_r(v)$. The lemma follows.

7. **Inclusion theorems for the N_{z_p} method.** In this section we show that the $B_{\alpha,\beta}$ and N_{q_p} methods can be obtained from the N_{z_p} method by specialization of the kernel $z_p(t)$ of that method.

THEOREM 7.1. *Every* N_{q_p} *kernel is an* N_{z_r} *kernel for* $r = 0, 1, 2, \cdots, p$.

Proof. Every N_{q_p} kernel, $p \geqq 1$, obviously satisfies Conditions (1) to (5) of Definition 6.1; and every N_{q_0} kernel is non-negative for $0 < t < 1$ and satisfies Condition 2, for $p = 0$. For the N_{q_p} method, the function

$$Q_p(\pi/v) \equiv \int_{1-\frac{\pi}{v}}^{1} (-1)^p t^p \frac{d^p}{dt^p} q_p(t) dt,$$

[15] Nevanlinna, *loc. cit.*[3]

which is a bounded monotone decreasing function of v for $v \geqq \pi$, approaches 0 as $v \to \infty$; and is such that

$$\lim_{\delta \to 0} \int_{\delta}^{1} Q_p(t)/t \, dt$$

exists; and may be taken as $Z_p(v)$.[16] Then

$$\int_{\pi}^{\infty} Z_p(v)/v \, dv = \int_{0}^{1} Z_p(\pi/v)/v \, dv$$

$$= \int_{0}^{1} Q_p(v)/v \, dv.$$

Hence every N_{q_p} kernel is an N_{z_p} kernel. We complete the proof by using Lemma 6.2.

THEOREM 7.2. *For C sufficiently large and either $\alpha = p$, $\beta > 1$ or $\alpha > p$, every $B_{\alpha,\beta}$ kernel is an N_{z_r} kernel, $r = 0, 1, 2, \cdots, p$.*

Proof. When $\alpha = p$, $\beta > 1$; $p < \alpha < p + 1$; or $\alpha = p + 1$, $\beta \leqq 0$; the theorem follows from Theorems 4.1 and 7.1. When $\alpha = p + 1$, $\beta > 0$, it follows from the definition of $B_{\alpha,\beta}(t)$ and Lemmas 3.2, 3.3, 3.4, and 3.6 that $B_{\alpha,\beta}(t)$ satisfies Conditions (1) to (5) of Definition 6.1, and that

$$\left| \int_{0}^{1} t^p \frac{d^p}{dt^p} B_{p+1,\beta}(t) \begin{matrix} \sin \\ \cos \end{matrix} vt \, dt \right| \leqq K/v.$$

The function K/v may be taken as $Z_p(v)$ for this case. Hence $B_{p+1,\beta}(t)$ is an N_{z_p} kernel. We complete the proof by using Lemma 6.2.

8. **The effectiveness of the N_{z_p} method for summing Fourier series.** The following theorem establishes for the N_{z_p} method all of the theorems concerning the summation of Fourier series which we give in our first and second papers.[17]

THEOREM 8.1. *Theorems concerning the effectiveness of the N_{z_p} method for the summation of Fourier series, conjugate Fourier series, and the derived series of such series, are obtained by replacing in the statements of Theorems* 6.1, 6.2, 6.3, 6.4, *and* 7.1 *of our first paper N_q by N_{z_p}, $p \geqq 0$, and by replacing in the statements of Theorems* 9.1, 9.2, 9.3, *and* 9.4 *of our second paper N_{q_0} by N_{z_p}, $p \geqq 0$; N_{q_1} by N_{z_p}, $p \geqq 1$; and N_{q_p} by N_{z_p}.*

Proof. The theorem can be established by replacing in Sections 4, and 6 of our first paper and in Sections 4, 6, 7, 8, and 9 of our second paper, for every specified p, $q_p(t)$ by $z_p(t)$, N_{q_p} by N_{z_p}, $Q_p(\pi/ns)$ by $Z_p(ns)$, and

$$\int_{0}^{1} Q_p(s)/s \, ds \quad \text{by} \quad \int_{\pi}^{\infty} Z_p(s)/s \, ds;$$

changing references to Lemmas 5.1, 5.2, and 5.3 to references to Condition (6) [Definition 6.1] of this paper; making minor changes in the notation and refer-

[16] Moursund, *loc. cit.*[2], §5.

[17] *Loc. cit.*[7] and *loc. cit.*[2].

ences suggested by the above changes; and then using Lemma 6.2. Some of the proofs can be shortened and simplified by the use of Lemma 6.2. The proof in Section 8 (second paper) that $L_3 = O(1)$ as $\delta \to 0$ must be replaced by

$$
\begin{aligned}
|\, L_3 \,| &= [o(1) \text{ as } \delta \to 0] \int_{\frac{\pi}{n}}^{\delta} 1/s \, ds \int_{\frac{\pi}{n s}}^{1} Z_0(nts) \, dt \\
&= [o(1) \text{ as } \delta \to 0] \int_{\delta}^{1} dt \int_{\frac{\pi}{n t}}^{1} Z_0(nts)/s \, ds \\
&= [o(1) \text{ as } \delta \to 0] \int_{\delta}^{1} dt \int_{\pi}^{nt} Z_0(v)/v \, dv \\
&\leqq [o(1) \text{ as } \delta \to 0] \int_{\pi}^{\infty} Z_0(v)/v \, dv = o(1) \text{ as } \delta \to 0.
\end{aligned}
$$

From theorems concerning the N_{z_p} method we obtain, as a consequence of Theorem 7.2, theorems for the $B_{\alpha,\beta}$ method in an even more simple manner than we did in §5. We do not even have to use the regularity-consistency property of the $B_{\alpha,\beta}$ method given by Lemma 2.2.

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ON THE APPROXIMATION OF LEBESGUE INTEGRALS BY RIEMANN SUMS

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1. By an investigation of the properties of the integral in a certain space of an infinite number of dimensions[1] I was incidentally led to the following theorem on ordinary Lebesgue integrals which does not seem to be known.

THEOREM A. *Let $f(x)$ be periodic with the period 1 and integrable in the Lebesgue sense. Then if we write*

$$f_n(x) = \frac{1}{n}\sum_{k=1}^{n} f\left(x + \frac{k}{n}\right)$$

the sequence of functions $f_{2^n}(x)$ converges for almost all x to the limit

$$A = \int_0^1 f(x)\,dx.$$[2]

The point is that the sequence 2^n is independent of $f(x)$; from the well known fact that the sequence $f_n(x)$ converges strongly to A in the sense that

$$\int_0^1 |\, f_n(x) - A \,|\, dx \to 0$$

follows in the direction of our theorem only the existence of sequences m_n such that $f_{m_n}(x)$ tends to A almost everywhere, but the sequences m_n found in this way will depend on $f(x)$.

The proof given below would prove more generally that $f_{m_n}(x)$ tends to A almost everywhere for any sequence m_n where m_{n-1} divides m_n. I could not decide whether the sequence $f_n(x)$ itself converges to A almost everywhere.

The approximation of Lebesgue integrals by Riemann sums has been studied by Hahn[3] but his results do not seem to have any connection with the above theorem.

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[1] An exposition of this theory and of some of its applications will appear in Acta mathematica.

[2] If we write $x = [x] + \frac{x_1}{2} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \cdots$ the function $f(x)$ becomes a function of the sequence of variables $x_1, x_2, x_3, \cdots$ each of which takes only the two values 0 and 1. The function $f_{2^n}(x)$ may be said to be the integral of $f(x)$ with respect to the variables $x_1, x_2, \cdots, x_n$; it is a function depending only on the variables $x_{n+1}, x_{n+2}, \cdots$. Theorem A may thus be looked upon as a natural extension of the theorem of Fubini to this special case of functions of an infinite number of variables.

[3] H. Hahn, Über Annäherung an Lebesguesche Integrale durch Riemannsche Summen. Wiener Sitzungsberichte **123** II a 1 (1914), 713–743.

2. In the proof of Theorem A we may clearly suppose that $f(x)$ is real.

Let us consider the function

$$\Phi(x) = \overline{\lim_{n\to\infty}} f_{2^n}(x).$$

If we can prove that $\Phi(x) \leqq A$ for almost all x the proof will be completed; for then replacing $f(x)$ by $-f(x)$ we obtain a corresponding result for the lower limit and both relations together give the theorem.

Now the function $f_{2^n}(x)$ is periodic with the period $1/2^n$. This implies that the function $\Phi(x)$ is periodic with the period $1/2^n$ for *any* n, and since a measurable function with arbitrarily small periods is always a constant almost everywhere we conclude that $\Phi(x) = C$, where C is some constant, for almost all x. In order to prove that $C \leqq A$ we first prove the following lemma.

LEMMA 1. *Let N and B be given and let E denote the set of points where*

$$\overline{\text{bound}}_{n=0,1,\cdots,N} \{f_{2^n}(x)\} \geqq B;$$

then

$$\int_E f(x)\,dx \geqq B \cdot mE.^4$$

Let E_n denote the set of points where $f_{2^n}(x) \geqq B$. Then we have clearly

$$E = E_N + E_N^* E_{N-1} + E_N^* E_{N-1}^* E_{N-2} + \cdots + E_N^* \cdots E_1^* E_0$$

denoting for the moment by E_n^* the complementary set of E_n. Let us write for shortness

$$A_n = E_N^* \cdots E_{n+1}^* E_n.$$

For any n the set E_n and therefore also the set E_n^* is periodic with period $1/2^n$; it follows that also the set A_n is periodic with this period. This implies that for any n

$$\int_{A_n} f(x)\,dx = \int_{A_n} f_{2^n}(x)\,dx \geqq B \cdot mA_n.$$

Adding these relations for $n = 0, 1, \cdots, N$ and using that the sets A_n have no common points we obtain the desired inequality.

Making $N \to \infty$ in Lemma 1 we obtain the following lemma.

LEMMA 2. *Let B be given and let E denote the set of points where*

$$\overline{\text{bound}}_{n=0,1,2,\cdots} \{f_{2^n}(x)\} \geqq B;$$

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[4] The set E is periodic with the period 1; by mE is meant the measure of a period of E; similarly the integral over E means the integral over a period. The same remark applies to the following considerations, all sets being periodic with the period 1.

then

$$\int_E f(x)\,dx \geqq B \cdot mE.$$

The proof of Theorem A is now immediate. In fact if we take $B < C$ the set E in Lemma 2 is the whole x-axes with the exception of a null-set. Consequently we get

$$A = \int_0^1 f(x)\,dx \geqq B \cdot 1 = B$$

and since this is true for any $B < C$ it follows that $C \leqq A$.

3. We allow again $f(x)$ to be complex valued.

Let us consider the smallest majorant of the sequence $f_{2^n}(x)$, that is the function

$$M(x) = \overline{\underset{n=0,1,2,\cdots}{\text{bound}}}\ \{|\, f_{2^n}(x)\,|\}.$$

This function has again the period 1. It is easy to construct an integrable function $f(x)$ with period 1 for which $M(x)$ is not integrable. A theorem of Hardy and Littlewood[5] suggests the problem what can be said about the integrability of $M(x)$ when $f(x)$ is known to belong to the class L^p where $p > 1$; the answer is as we should expect that in this case $M(x)$ also belongs to L^p. We have the following precise theorem.

THEOREM B. *Let $f(x)$ belong to L^p where $p > 1$; then the same is true for $M(x)$ and we have*

$$\int_0^1 M(x)^p\,dx < \left(\frac{p}{p-1}\right)^p \int_0^1 |\, f(x)\,|^p\,dx$$

except when $f(x)$ is null.

I could not decide whether the constant in this theorem is the best possible constant.

We may suppose in the proof that $f(x)$ is real and non-negative; for if not we may replace $f(x)$ by $|\, f(x)\,|$ and this change does not decrease $M(x)$. We know from Lemma 2 that if B is given and E denotes the set where $M(x) \geqq B$, then

$$\int_E f(x)\,dx \geqq B \cdot mE;$$

thus Theorem B will follow from the following lemma.

LEMMA 3. *Let $f(x)$ and $M(x)$ be non-negative functions such that the last mentioned inequality is true for any B when E denotes the set where $M(x) \geqq B$. Then if $f(x)$ belongs to L^p where $p > 1$ the same is true for $M(x)$ and we have*

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[5] G. H. Hardy and J. E. Littlewood, A maximal theorem with function-theoretic applications. Acta mathematica **54** (1930), 81–116.

$$\int_0^1 M(x)^p\,dx < \left(\frac{p}{p-1}\right)^p \int_0^1 f(x)^p\,dx$$

except when $f(x)$ is null.

There is no need for giving the proof of this last lemma here. It is obtained from the last part of the proof by F. Riesz[6] for the main theorem of the paper of Hardy and Littlewood referred to above by combination with an earlier inequality of Hardy.

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[6] F. Riesz, Sur un théorème de maximum de MM. Hardy et Littlewood. Journal of the London mathematical society 7 (1932), 10–13.

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DISTRIBUTION FUNCTIONS AND POSITIVE-DEFINITE FUNCTIONS

BY S. BOCHNER AND B. JESSEN

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Introduction

The distribution of the values of a real almost periodic function or more generally of an almost periodic curve in several dimensions has been investigated lately by Wintner, Haviland and others.[1] One of the methods applied by Wintner depends on the use of Fourier transforms and is as such a standard method in the theory of probability. The object of the present paper is to point out that the established results can be deduced simply, and in a general form, from general theorems concerning Fourier transforms without introducing moment arguments.

1. Positive-Definite Functions[2]

We shall consider distribution functions in a Euclidean space $\mathfrak{R}$ of k dimensions. As the number of dimensions is of no avail it is convenient to have a notation independent of it. A point x, α, $\cdots$ or a vector function $f(t)$ will always be understood to consist of k components. Naturally sum or difference of two points is to be taken in the vector sense; the formal products $x\alpha$, $xf(t)$, $\cdots$ chiefly occurring in the exponentials $e^{ix\alpha}$, $e^{ixf(t)}$, $\cdots$ will be the scalar inner product. An inequality between any two points means the corresponding inequality in all components; for instance $f(t) < \alpha$ means that each component of $f(t)$ is smaller than the corresponding component of α.

A *distribution function* will be an additive, non-negative interval function $V(\mathfrak{I})$ defined for all bounded and unbounded intervals in $\mathfrak{R}$. Its total variation is the value $V(\mathfrak{R})$. It may be derived in the usual way as a k-fold difference from a monotone point function $V(\alpha)$, which is defined for any point α by

$$V(\alpha) = V(\mathfrak{I}_\alpha)$$

where $\mathfrak{I}_\alpha$ denotes the interval $\beta < \alpha$. $V(\mathfrak{I})$ and $V(\alpha)$ will denote the same function.

Two interval functions are called *essentially equal* if they are equal in their (bounded) intervals of continuity. A sequence of functions $V_n(\mathfrak{I})$ is *essentially convergent* to an interval function $V(\mathfrak{I})$ if the relation

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[1] See A. Wintner, Upon a statistical method in the theory of diophantine approximations. Amer. J. Math. **55** (1933), 309–331, where references are given.

[2] S. Bochner, Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse. Math. Ann. **108** (1933), 378–410.

$$V(\mathfrak{I}) = \lim_{n\to\infty} V_n(\mathfrak{I}) \tag{1}$$

holds for all (bounded) intervals in which $V(\mathfrak{I})$ is continuous. The sequence is called *strictly convergent* if the limit relation (1) also holds for $\mathfrak{I} = \mathfrak{R}$.

A complex-valued function $f(x)$ in $\mathfrak{R}$ is called *positive definite* if it is continuous[3] and if for any points $x_1, x_2, \cdots, x_m (m = 2, 3, 4, \cdots)$ and any complex numbers $\zeta_1, \zeta_2, \cdots, \zeta_m$ we have

$$\sum_{\mu=1}^{m}\sum_{\nu=1}^{m} f(x_\mu - x_\nu)\zeta_\mu\bar{\zeta}_\nu \geqq 0.$$

We shall denote the class of these functions by $\mathfrak{P}$. They have the following properties:[4]

I. In order that a function $f(x)$ belong to $\mathfrak{P}$ it is necessary and sufficient that there exists a distribution function $V(\alpha)$ such that

$$f(x) = \int e^{ix\alpha}\, dV(\alpha). \tag{2}$$

The integral is a Stieltjes integral over the whole $\mathfrak{R}$. The function $V(\alpha)$ in (2) is essentially unique; we shall call it the *transform* of $f(x)$.

II. If a sequence $V_n(\alpha)$ is strictly convergent to a function $V(\alpha)$ then the corresponding functions $f_n(x) \subset \mathfrak{P}$ are convergent, uniformly in $\mathfrak{R}$, to the corresponding function $f(x) \subset \mathfrak{P}$.

III. Conversely, if a sequence of functions $f_n(x) \subset \mathfrak{P}$ converge almost everywhere and for $x = 0$ to a function $f_0(x)$, then there exists a function $f(x) \subset \mathfrak{P}$ which coincides with $f_0(x)$ almost everywhere. And the transform of $f_n(x)$ converges essentially towards the transform of $f(x)$.

IV. In particular, if $f_n(x) \subset \mathfrak{P}$ converge almost everywhere and for $x = 0$ to a function $f_0(x)$ which is continuous in the point $x = 0$, then the transforms $V_n(\alpha)$ are *strictly convergent* towards the transform $V(\alpha)$ of a function $f(x) \subset \mathfrak{P}$ which in $x = 0$ and almost everywhere coincides with $f_0(x)$.

V. The product $f(x)\, g(x)$ of two positive-definite functions $f(x)$ and $g(x)$ is again positive-definite and the transforms $U(\alpha)$, $V(\alpha)$ and $W(\alpha)$ of $f(x)\, g(x)$, $f(x)$ and $g(x)$ are connected by the relations

$$U(\alpha) = \int V(\alpha - \beta)\, dW(\beta) \equiv \int W(\alpha - \beta)\, dV(\beta).$$

In other words, $U(\alpha)$ is the convolution of $V(\alpha)$ and $W(\alpha)$.

2. Distribution Functions

Let D be a set of finite measure in an appropriate space; the variable point in this space shall be denoted by t. Let $f(t)$ be a vector function whose components

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[3] Compare also F. Riesz, Über Sätze von Stone und Bochner. Acta Szeged **6** (1933), 184–198.

[4] loc. cit.[2]. Some of the results are here enunciated in a slightly more general form than originally. We shall not use property I; for the following it would be sufficient to define the functions of the class $\mathfrak{P}$ by the integral (2).

are defined and are measurable on D. We consider the distribution function

$$V_D(\alpha) = \frac{m(f(t) < \alpha)_D}{mD},$$

where $(f(t) < \alpha)_D$ denotes the set of points belonging to D for which $f(t) < \alpha$. By the definition of the Lebesgue integral we have

$$\frac{1}{mD}\int_D e^{ixf(t)}\,dt = \int e^{ix\alpha}\,dV_D(\alpha); \tag{3}$$

thus $V_D(\alpha)$ is the transform of the positive definite function to the left.

Now let $\{D\}$ denote a system of sets D, each of which has a finite measure; we suppose that there exist sequences $\{D_n\}$ in $\{D\}$ such that $mD_n \to \infty$; each such sequence shall be called an *admissible sequence*. If our vector function $f(t)$ is defined on the system $\{D\}$ we shall say that it has a *distribution on* $\{D\}$ if there exists a distribution function $V(\alpha)$ such that for each admissible sequence $\{D_n\}$ the sequence $V_{D_n}(\alpha)$ is *strictly convergent* towards $V(\alpha)$. Of course the total variation of $V(\alpha)$ is then also 1.

In order to extend the formula (3) we have to introduce the notion of a mean value for a complex valued function $g(t)$ defined on $\{D\}$; we say that $g(t)$ has a *mean value* on $\{D\}$ if $g(t)$ is integrable over each D and if there exists a number $M\{g(t)\}$ such that for any admissible sequence $\{D_n\}$ we have

$$M\{g(t)\} = \lim_{n\to\infty}\frac{1}{mD_n}\int_{D_n} g(t)dt.$$

With these definitions the formula (3) will generalize into the formula

$$M\{e^{ixf(t)}\} = \int e^{ix\alpha}\,dV(\alpha); \tag{4}$$

with regard to its validity we have in consequence of §1 the following theorem.

THEOREM 1. *If the vector function $f(t)$ has a distribution $V(\alpha)$ on $\{D\}$ then for each x the mean value $M\{e^{ixf(t)}\}$ exists and we have formula* (4). *Moreover for each admissible sequence $\{D_n\}$ the limit*

$$\lim_{n\to\infty}\frac{1}{mD_n}\int_{D_n} e^{ixf(t)}\,dt \tag{5}$$

exists uniformly in x.

Conversely, if for each admissible sequence $\{D_n\}$ the limit (5) *exists for almost all x and for $x = 0$ and is continuous in the point $x = 0$ then $f(t)$ has a distribution, and consequently we have the formula* (4).

It will be useful also to have a theorem asserting the existence of a distribution of a vector function $f(t)$ which in a certain generalized sense is the limit of functions $f_m(t)$ each of which has a distribution on $\{D\}$. We have immediately the following theorem.

THEOREM 2. *Let $f_m(t)$ be a sequence of vector functions each of which has a distribution $V_m(\alpha)$ on $\{D\}$ and let $f(t)$ be a measurable function on $\{D\}$ such that for each admissible sequence $\{D_n\}$ the upper limit*

$$\varphi(m, x) = \overline{\lim_{n\to\infty}} \left| \frac{1}{mD_n} \int_{D_n} (e^{ixf(t)} - e^{ixf_m(t)})\, dt \right|$$

converges to zero for almost all x and for $x = 0$ and continuously in the point $x = 0$[5] as $m \to \infty$. Then the function $f(t)$ has also a distribution $V(\alpha)$, and $V(\alpha)$ is the strict limit of $V_m(\alpha)$.

The condition regarding $\varphi(m, x)$ is satisfied if for each admissible sequence $\{D_n\}$ the number

$$\psi(m) = \overline{\lim_{n\to\infty}} \frac{1}{mD_n} \int_{D_n} |f(t) - f_m(t)|^p\, dt \quad (p > 0)$$

where $|f(t) - f_m(t)|$ denotes the length of the vector $f(t) - f_m(t)$ tends to zero as $m \to \infty$. In this case we shall say that the sequence $f_m(t)$ converges strongly to $f(t)$ on $\{D\}$ with exponent p.

3. Application

Let t denote a real variable, $-\infty < t < \infty$, and let $\{D\}$ denote either the system of all finite intervals $a < t < b$ or the system of all symmetric intervals $-T < t < T$. A distribution with respect to the second system if it exists will be called a distribution over $-\infty < t < \infty$; if it exists with respect to the first system it will be called a uniform distribution over $-\infty < t < \infty$.

Let $f(t)$ be a vector function whose components are almost periodic in the sense of Bohr. Since the functions $e^{ixf(t)}$ form in every bounded x-interval a majorisable class of almost periodic functions the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_a^{a+T} e^{ixf(t)}\, dt$$

exists uniformly in a and uniformly in every bounded x-interval. From Theorem 1 we therefore conclude.

THEOREM 3A. *Any vector function $f(t)$ whose components are almost periodic in the sense of Bohr possesses a uniform distribution over $-\infty < t < \infty$ defined by* (4).

A vector function $f(t)$ is almost periodic in the sense of Weyl (with exponent 1)[6] if there exists a sequence of ordinary almost periodic vector functions $f_m(t)$ such that the number

$$\chi(m) = \lim_{T\to\infty} \overline{\text{bound}}_{-\infty<a<\infty} \frac{1}{T} \int_a^{a+T} |f(t) - f_m(t)|\, dt$$

tends to zero as $m \to \infty$. From Theorem 2 we therefore conclude

[5] That is $\varphi(m, x_m) \to 0$ whenever $x_m \to 0$.

[6] We refer to the book of Besicovitch, Almost periodic functions. Cambridge 1932.

THEOREM 3B. *Any vector function $f(t)$ whose components are almost periodic in the sense of Weyl possesses also a uniform distribution over $-\infty < t < \infty$ defined by* (4).

A vector function $f(t)$ is almost periodic in the sense of Besicovitch (with exponent 1) if there exists a sequence of ordinary almost periodic vector functions $f_m(t)$ such that the number

$$\chi(m) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} |f(t) - f_m(t)|\, dt \tag{6}$$

tends to zero as $m \to \infty$. We therefore have:

THEOREM 3C. *Any vector function $f(t)$ whose components are almost periodic in the sense of Besicovitch possesses a distribution over $-\infty < t < \infty$ defined by* (4).

4. Independent Distributions

In this section every vector function $f(t)$ will be almost periodic in the sense of Besicovitch (with exponent 1) and $f(t)$ will be approximated to by the sequence $f_m(t)$ if the number (6) tends to zero.

We define the modulus M_f of $f(t)$ as the smallest modulus containing the Fourier exponents of the components of $f(t)$. We call a system of vector functions $f(t)$ *independent* if the moduli of any two of them have only the number 0 in common.

THEOREM 4. *Let $f(t)$ and $g(t)$ be independent vector functions. Then we have*

$$M\{e^{ix(f(t)+g(t))}\} = M\{e^{ixf(t)}\}\, M\{e^{ixg(t)}\} \tag{7}$$

and consequently the distribution $U(\alpha)$ of $f(t) + g(t)$ is the convolution of the distributions $V(\alpha)$ and $W(\alpha)$ of $f(t)$ and $g(t)$.

The relation (7) is obvious if the vector functions $f(t)$ and $g(t)$ are almost periodic in the sense of Bohr. In the general case we approximate to $f(t)$ and $g(t)$ by ordinary almost periodic vector functions $f_m(t)$ and $g_m(t)$ without introducing new exponents.

A simple oscillation $f(t) = ae^{i\lambda t} = |a|\, e^{i(\lambda t+\rho)}$ is the two dimensional curve with the components $(|a| \cos(\lambda t + \rho), |a| \sin(\lambda t + \rho))$. The corresponding mean value $M\{e^{ixf(t)}\}$, where x is $(|x| \cos\sigma, |x| \sin\sigma)$, is

$$M\{e^{ixf(t)}\} = M\{e^{i|xa|\cos(\lambda t+\rho-\sigma)}\} = J_0(|xa|)$$

where $J_0(y)$ denotes the Bessel function of order zero. Combining this with Theorems 2 and 4 we obtain

THEOREM 5A. *If $f(t) \sim \sum_\lambda a_\lambda e^{i\lambda t}$ is an ordinary almost periodic function with linearly independent exponents (and consequently $\sum_\lambda |a_\lambda|$ is finite) then its distribution $V(\alpha)$ is determined by*

$$\int e^{ix\alpha}\, dV(\alpha) = \prod_{\lambda} J_0(|\, xa_\lambda\,|). \tag{8}$$

If $f(t) \sim \sum_{\lambda} a_\lambda e^{i\lambda t}$ is almost periodic in the Besicovitch sense then there exists a system of coefficients $r_\lambda^{(m)}$ defined for the given λ's and for $m = 1, 2, 3, \cdots$ with the following properties: 1) For a given value of m only a finite number of the $r_\lambda^{(m)}$'s differ from zero. 2) $0 \leqq r_\lambda^{(m)} \leqq 1$ and $\lim_{m\to\infty} r_\lambda^{(m)} = 1$ for all λ. 3) The exponential polynomials $S_m(t) = \sum_{\lambda} r_\lambda^{(m)} a_\lambda e^{i\lambda t}$ approximate $f(t)$. If the exponents λ are independent we have by Theorem 5A for the distribution function $V_m(\alpha)$ of $S_m(\alpha)$

$$\int e^{ix\alpha}\, dV_m(\alpha) = \prod_{\lambda} J_0(|\, x\, r_\lambda^{(m)} a_\lambda\,|). \tag{9}$$

By Theorem 3c the distribution $V_m(\alpha)$ converges strictly to the distribution $V(\alpha)$ of $f(t)$. Therefore the right-hand side of the relation (9) must converge uniformly for all x. Since the power series of the Bessel function $J_0(y)$ begins with $1 - \frac{y^2}{4}$ and since $|J_0(y)| < k_0 < 1$ if $|y| > y_0 > 0$ this is only possible when the sum $\sum_{\lambda} |a_\lambda|^2$ is finite (for otherwise the product to the right in (9) would converge to 1 for $x = 0$ and to 0 for $x \neq 0$). We therefore conclude:

THEOREM 5B. *If $f(t) \sim \sum_{\lambda} a_\lambda e^{i\lambda t}$ is an almost periodic function in the Besicovitch sense with linearly independent exponents the sum $\sum_{\lambda} |a_\lambda|^2$ is finite and the distribution function $V(\alpha)$ of $f(t)$ satisfies the relation* (8).

Theorem 5a is due to Wintner.[7] His further results concerning the smoothness of the function $V(\alpha)$ are deducible from the relation (8) and apply also to Besicovitch functions.

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[7] Loc. cit.[1]

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EXPONENTIAL POLYNOMIALS

BY E. T. BELL

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The exponential numbers considered in another paper[1] suggested the extensive generalizations to the polynomials of the present paper. The term "exponential polynomial" here refers to polynomials generated by operating on exponential functions $\exp(f(x))$ by differentiation, or by expanding the exponential into a power series in x.

The polynomials ξ(§§1–3) are one of the many possible generalizations of Hermite's polynomials. From the ξ are constructed functions $\zeta(t)$ which are orthogonal in the interval $-\infty \leqq t \leqq +\infty$. With respect to linear differential equations the Hermite polynomials are a sort of singular case of the ξ, see §2.

The polynomials ϕ generalize the ξ, and are a generalization of Appell polynomials; see §§4–6, 10. Closely connected with the ϕ are the polynomials Y (§§7–9) which generalize the ξ and (in a sense which will be clear from (7.16)), also the ϕ. They are introduced in connection with a certain determinant in (6.5) on which the arithmetical properties of the ϕ depend. The Y also give rise to a set of orthogonal functions. Neither ϕ_n nor Y_n satisfies any differential equation whose order is independent of n. The number of terms in ϕ_n is the total number of partitions of n; the sum of all the coefficients is the positive integer ϵ_n, and the ϵ's are one of the sequences whose congruence properties were discussed in the paper mentioned.

The polynomials ψ (§§11, 12) are a generalization, in another direction, of Appell polynomials. The possible types of differential equations which, under suitable hypotheses, the ψ can satisfy, are determined, and the degeneration in a special case to Appell's equations is noted.

All of the polynomials have interesting arithmetical properties, some of which are discussed.

I. The polynomials ξ

1. Let r denote a constant integer > 0, n an arbitrary integer $\geqq 0$, and x, t independent variables; write $D_t \equiv d/dt$. The polynomials $\xi_n \equiv \xi_n(x, t; r)$ are defined by $\xi_{-n-1} = 0$, and

$$\xi_n \equiv \exp(-xt^r) D_t^n (\exp(xt^r)). \tag{1.1}$$

[1] To be published in 1934; see the abstract in the *Bulletin of the American Mathematical Society*, vol. 39, 1933, p. 667.

For $r = 2$, the ξ_n are Hermite polynomials; for $r > 2$ the ξ_n do not seem to have been discussed, although Hoppe[2] gave the equivalent of the explicit form (1.6) as an illustration of his method of obtaining higher derivatives without, however, investigating any of the properties of the polynomials.

Operation with D_t on both sides of (1.1) gives

$$\xi_{n+1} = rxt^{r-1} + D_t\xi_n, \tag{1.2}$$

from which it follows by mathematical induction that

$$t^n\xi_n = \sum_{i=0}^{n} c_i(n)\ (xt^r)^{n-i}, \tag{1.3}$$

where $c_i(n)$ is a polynomial of degree n in r with integer coefficients, and

$$c_0(0) = 1, \qquad c_0(n+1) = rc_0(n), \tag{1.4}$$

$$c_j(n+1) = rc_j(n) - [n - r(n-j+1)]c_{j-1}(n) \qquad (0 < j < n+1).$$

The c's can be calculated successively from (1.4); thus

$$c_0(n) = r^n, \qquad c_1(n) = \binom{n}{2}(r-1)r^{n-1}, \cdots;$$

but it is shorter to obtain the explicit forms otherwise, as in (1.6). For numerical checks we write down the following from $\xi_0 = 1$ and (1.2).

$$\xi_1 = rxt^{r-1},$$
$$\xi_2 = r^2x^2t^{2r-2} + r(r-1)xt^{r-2},$$
$$\xi_3 = r^3x^3t^{3r-3} + 3r^2(r-1)x^2t^{2r-3} + r(r-1)(r-2)xt^{r-3}.$$

From (1.3),

$$\xi_n = \sum_{i=0} c_i(n)\, x^{n-i}\, t^{(n-i)r-n}, \tag{1.5}$$

where the sum continues so long as $n - i \geqq 0$, $(n-i)r - n \geqq 0$. Hence, if $[y]$ denotes the greatest integer in y, the upper limit of the summation is $[n\,(r-1)/r]$.

To obtain the explicit form, multiply

$$D_t^n \exp(xt^r) = n! \sum_{m \geq [n/r]}^{\infty} \frac{1}{m!}\binom{rm}{n} x^m t^{rm-n}$$

[2] R. Hoppe, *Journal für Mathematik*, vol. 33, 1846, pp. 76–79. I owe this reference to Professor Bateman, who has given an alternative expression for the ξ_n involving Stirling polynomials, which follows readily from (1.6) here. Professor Bateman has also discussed the polynomials $\exp(-xt^r)D_t^n[x^{sr}\exp(xt^r)]$, where s is an interger > 0.

by the power series for $\exp(-xt^r)$; thus

$$\xi_n = n!\sum_{h=0}^{\infty}\frac{x^h t^{rh-n}}{h!}\sum_{s=0}^{\leqq h-1}(-1)^s\binom{h}{s}\binom{r(h-s)}{n}.$$

The last binomial coefficient is zero unless $r(h-s)\geqq n$, and by (1.5) the degree in x is n. Hence finally,

$$\xi_n = n!\sum_{h=a}^{n}\frac{x^h t^{rh-n}}{h!}\sum_{s=0}^{b}(-1)^s\binom{h}{s}\binom{r(h-s)}{n}, \tag{1.6}$$

$$a \equiv n - [n(r-1)/r], \qquad b = [(rh-n)/r].$$

It follows incidentally that

$$\sum_{s=0}^{b}(-1)^s\binom{h}{s}\binom{r(h-s)}{n} = 0, \qquad h > n,$$

which is a property of the Stirling polynomials.

The generating function for ξ_n follows from (1.1) by Taylor's theorem,

$$\exp[x((h+t)^r - t^r)] = \exp(h\xi), \tag{1.7}$$

where the exponential on the right is $\exp(h\xi) \equiv \xi_0 + h\xi_1 + h^2\xi_2/2! + \cdots$, as usual in the symbolic or umbral calculus of Blissard.[3]

Operating on (1.7) with D_h, D_t, D_x respectively, we get

$$\xi_{n+1} = n!\,rx\sum_{s=0}^{r-1}\binom{r-1}{s}t^s\frac{\xi_{n-r+s+1}}{(n-r+s+1)!},$$

$$D_t\xi_n = n!\,rx\sum_{s=0}^{r-2}\binom{r-1}{s}t^s\frac{\xi_{n-r+s+1}}{(n-r+s+1)!}, \qquad r > 1,$$

$$D_x\xi_n = n!\sum_{s=0}^{r-1}\binom{r}{s}t^s\frac{\xi_{n-r+s}}{(n-r+s)!},$$

in which (by the notation explained) $\xi_j \equiv \xi_j(x, t; r)$, and the sums either continue

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[3] The account in Lucas' *Théorie des Nombres*, Chap. 13, may be supplemented by my own extensions in *Algebraic Arithmetic*, 1927. Umbral derivatives, as in

$$D_\xi \exp(h\xi) = \xi \exp(h\xi)$$

satisfy the formal laws of differentiation; all indicated operations upon umbrae are to be performed as in common algebra before exponents are lowered, and if a, b are ordinaries such that $ab \neq 0$, and α, β umbrae, the value of

$$(a\alpha + b\beta)^0 \text{ is } a^0\alpha^0 b^0\beta^0, = \alpha_0\beta_0.$$

The last avoids exceptions in the manipulations and results. The formal use of infinite processes in this connection is justified in the work cited.

to the upper limits shown, or terminate with the largest s which makes the suffix of ξ non-negative. These may be written

$$\xi_{n+1} = rx \sum_{s=0}^{r-1} (r-s-1)! \binom{r-1}{s} \binom{n}{r-s-1} t^s \xi_{n-r+s+1}, \tag{1.8}$$

$$D_t \xi_n = rx \sum_{s=0}^{r-2} (r-s-1)! \binom{r-1}{s} \binom{n}{r-s-1} t^s \xi_{n-r+s+1}, \qquad r > 1, \tag{1.9}$$

$$D_x \xi_n = \sum_{s=0}^{r-1} (r-s)! \binom{r}{s} \binom{n}{r-s} t^s \xi_{n-r+s}. \tag{1.10}$$

We may write (1.7) more symmetrically as

$$\exp [x(h^r - t^r)] = \exp [(h-t)\xi], \tag{1.11}$$

from which (1.6) follows by equating coefficients of h^n. We may take (1.11) as the definition of the polynomials $\xi_n \equiv \xi_n(x, t; r)$, since (1.8), (1.9), which follow from (1.7), and hence from (1.11), imply (1.2). Replacing x by $x + y$ in (1.11) we have the addition theorem

$$\xi_n(x+y, t; r) = [\xi(x, t; r) + \xi(y, t; r)]^n, \tag{1.12}$$

and hence

$$\sum_{s=0}^{n} \binom{n}{s} \xi_{n-s}(x, t; r)\, \xi_s(-x, t; r) = \delta_{0n}, \tag{1.13}$$

where $\delta_{00} = 1$, $\delta_{0n} = 0$ $(n > 0)$. In (1.7) replace h, t by ch, ct, where c is independent of h. Then

$$\xi_n(c^r x, t; r) = c^n \xi_n(x, ct; r). \tag{1.14}$$

Substitution of t^{-1} for t in (1.7) gives

$$\xi_n(x, t^{-1}; r) = t^n \xi_n(xt^{-r}, 1; r). \tag{1.15}$$

Similarly we have

$$\xi_{sr}(x, 0; r) = \frac{(sr)!}{s!} x^s; \quad \xi_n(x, 0; r) = 0, \quad n \not\equiv 0 \bmod r. \tag{1.16}$$

From (1.14),

$$\xi_n(c^r x, t; r) + a_1 \xi_{n-1}(c^r x, t; r) + \cdots + a_n \xi_0(c^r x, t; r) \equiv 0, \tag{1.17}$$

identically in x, t, where c is any root of $c^n + a_1 c^{n-1} + \cdots + a_n = 0$. This is equivalent, by (1.6), to a set of identities in Stirling polynomials.

2. From (1.6), (1.8), (1.9) we readily find that

$$D_t \xi_n(x, t; 1) = 0,$$

$$D_t^2 \xi_n(x, t; 2) + 2xt D_t \xi_n(x, t; 2) - 2nx \xi_n(x, t; 2) = 0.$$

the second of which is Hermite's equation when $2x = -1$. Using (1.10) we get

$$xD_x\xi_n(x, t; 1) - n\xi_n(x, t; 1) = 0.$$

It will now be shown that if $r > 2$, *the polynomials* $\xi_n(x, t; r) \equiv \xi_n$ *satisfy no linear equation of constant order* $m > 0$ *of the form*

$$P_0D_t^m\xi_n + P_1D_t^{m-1}\xi_n + \cdots + P_m\xi_n = P_{m+1}, \tag{2.1}$$

where the P*'s are polynomials in* r, n, x, t *alone* (the case where some or all P's are constants included), *and if* $r > 1$, *the* ξ_n *satisfy no equation of the form*

$$P_0D_x^m\xi_n + P_1D_x^{m-1}\xi_n + \cdots + P_m\xi_n = P_{m+1}. \tag{2.2}$$

Thus, with respect to the property (2.1), x^n and the Hermite polynomials (with the parameter x) are unique among the $\xi_n(x, t; r)$, and x^n is unique with respect to (2.2).

Let r be > 1, $m > 0$. By repeated application of (1.9) it follows that $D_t^m\xi_n$, considered as a function of ξ's, is linear in $\xi_{n-j}(j = m, m+1, \cdots, m(r-1))$ alone, where account is taken of $\xi_{-s} = 0, s > 0$, say $D_t^m\xi_n = \lambda_m(\xi_{n-m}, \xi_{n-m-1}, \cdots, \xi_{n-m(r-1)})$, and that the coefficients of the ξ's in λ_m are polynomials in r, n, x, t alone. Similarly, from (1.8),

$$D_t^0\xi_n \equiv \xi_n = \lambda_0(\xi_{n-1}, \xi_{n-2}, \cdots, \xi_{n-r}),$$

where λ_0 is a linear function of the same kind as λ_m. Hence

$$\begin{aligned} D_t^0\xi_n &= \lambda_0(\xi_{n-1}, \xi_{n-2}, \cdots, \xi_{n-r}), \\ D_t^1\xi_n &= \lambda_1(\xi_{n-1}, \xi_{n-2}, \cdots, \xi_{n-(r-1)}), \\ &\cdots \\ D_t^m\xi_n &= \lambda_m(\xi_{n-1}, \xi_{n-2}, \cdots, \xi_{n-m(r-1)}). \end{aligned} \tag{2.3}$$

Since m, r are constant integers > 0, we can choose n so that $n - m(r-1) \geqq 0$. If (2.1) holds, it must be possible to eliminate $\xi_{n-1}, \xi_{n-2}, \cdots, \xi_{n-m(r-1)}$ from the $(m+1)$ equations (2.3). But these $m(r-1)$ ξ's are distinct. Hence, in order that the elimination be always possible, it is necessary that $m(r-1) = m$, or $m(r-2) = 0$. But $m > 0$; hence $r = 2$. Similarly for (2.2).

3. In this section we show that if

$$\zeta_n \equiv \zeta_n(x, t; 2r) \equiv e^{-xt^{2r}/2}\,\xi_n(-x, t; 2r), \qquad x > 0, \tag{3.1}$$

then the ζ_n form an orthogonal set in the interval $-\infty \leqq t \leqq +\infty$,

$$\int_{-\infty}^{\infty} \zeta_m\zeta_n\,dt = 0, \qquad m \neq n. \tag{3.2}$$

For the moment let x, the integer $r > 0$, and $f(t) \equiv f$ be unrestricted beyond the conditions

$$[e^{-xt^r}D_t^s f(t)]_{t=\pm\infty} = 0 \ (s = 0, \cdots, n-1). \tag{3.3}$$

Then

$$[(e^{-xt^r}D_t^s f)\xi_n(-x, t; r)]_{t=\pm\infty} = 0,$$

and hence, by repeated partial integration of the left of (3.4),

$$\int_{-\infty}^{\infty} f(t)\ (D_t^n e^{-xt^r})\ dt = (-1)^n \int_{-\infty}^{\infty} e^{-xt^r}(D_t^n f)\ dt. \tag{3.4}$$

Thus if we choose $f(t) \equiv P(t)$, where P is a polynomial of degree $< n$ in t, we have

$$\int_{-\infty}^{\infty} e^{-xt^r}P(t)\xi_n(-x, t; r)\ dt = 0. \tag{3.5}$$

The conditions (3.3) with $f \equiv P$ are satisfied when $x > 0$ and r is an even integer > 0. Hence (3.2) follows from (3.5) when we take $m < n$.

II. The Polynomials ϕ, Y

4. The ϕ_n are a generalization of the ξ_n; see (4.51). Let $\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots$ be an infinite sequence of independent variables. The case when some or all of the α are absolute constants is included; in particular, α_n may be zero for all $n > m$, where m is a constant integer. The polynomials

$$\phi_n \equiv \phi_n(\alpha) \equiv \phi_n(\alpha_1, \cdots, \alpha_n) \qquad (n = 0, 1, \cdots)$$

are defined by

$$\phi_0 = 1, \qquad \phi_{-n-1} = 0, \qquad \phi^{n+1} = \alpha(\phi + \alpha)^n, \qquad n \geqq 0. \tag{4.1}$$

In ordinary notation the last is

$$\phi_{n+1} = \sum_{s=0}^{n} \binom{n}{s} \alpha_{s+1}\phi_{n-s}. \tag{4.2}$$

For verifications of formulas we give the first 6 ϕ's

$$\phi_0 = 1, \quad \phi_1 = \alpha_1, \quad \phi_2 = \alpha_1^2 + \alpha_2, \quad \phi_3 = \alpha_1^3 + 3\alpha_1\alpha_2 + \alpha_3,$$
$$\phi_4 = \alpha_1^4 + 6\alpha_1^2\alpha_2 + 4\alpha_1\alpha_3 + 3\alpha_2^2 + \alpha_4,$$
$$\phi_5 = \alpha_1^5 + 10\alpha_1^3\alpha_2 + 10\alpha_1^2\alpha_3 + 15\alpha_1\alpha_2^2 + 5\alpha_1\alpha_4 + 10\alpha_2\alpha_3 + \alpha_5.$$

Let t be a parameter. Then, writing

$$f(t) \equiv \exp(t\alpha) \equiv \alpha_0 + \alpha_1 t + \alpha_2 t^2/2\,! + \cdots,$$

we have the generating identity of the ϕ,

$$\exp[f(t) - \alpha_0] = \exp(t\phi) \tag{4.3}$$

For, operation with D_t on (4.3) gives

$$\alpha \exp[(\alpha + \phi)t] = \phi \exp(t\phi),$$

and hence (4.1) by comparing coefficients of t^n. Conversely, (4.1) implies (4.3). By (4.1) the coefficients in ϕ_n are positive integers, and it follows by mathematical induction from (4.1) that

$$\phi_n = \alpha_1^n + \alpha_n + P(\alpha_1, \cdots, \alpha_{n-1}), \qquad n > 1, \tag{4.4}$$

where $P(\alpha_1, \cdots, \alpha_{n-1})$ is a polynomial in $\alpha_1, \cdots, \alpha_{n-1}$ alone and is of degree $< n$.

From (4.3),

$$\exp(t\alpha_1 + t^2\alpha_2/2! + \cdots) = \phi_0 + t\phi_1 + t^2\phi_2/2! + \cdots, \tag{4.41}$$

and hence we have the explicit form

$$\phi_n = n! \sum \frac{\alpha_1^{s_1}\alpha_2^{s_2} \cdots \alpha_n^{s_n}}{(1!)^{s_1}(2!)^{s_2} \cdots (n!)^{s_n}.\ \ s_1!\, s_2! \cdots s_n!} \quad (n > 0), \tag{4.5}$$

where the sum refers to all integers $s_1, \cdots, s_n \geqq 0$ such that $n = s_1 + 2s_2 + \cdots + ns_n$. Hence *the number π_n of terms in ϕ_n is equal to the total number of partitions of n.* Thus the complexity increases rapidly with n; for example, $\pi_6 = 11$, $\pi_{22} = 1002$.

Comparing (1.7), (4.41), we see that for the particular sequence α indicated in (4.51) we have

$$\phi_n(\alpha) = \xi_n(x, t; r),\ \alpha_j = j!\binom{r}{j}xt^{r-j}(j = 1, \cdots, r);\ \alpha_n = 0, \quad n > r. \tag{4.51}$$

Another interesting sequence of positive integers connected with ϕ are the ϵ, defined by

$$\epsilon_n \equiv \phi_n(1, \cdots, 1), \qquad n > 0; \qquad \epsilon_0 = 1; \tag{4.6}$$

ϵ_n is the sum of the coefficients of ϕ_n. The recurrence for the ϵ_n is, from (4.1) with all α's replaced by 1,

$$\epsilon_{n+1} = (\epsilon + 1)^n, \qquad \epsilon_0 = 1, \tag{4.7}$$

and it is readily shown[4] that

$$\epsilon_n = \left(\frac{\Delta}{1!} + \frac{\Delta^2}{2!} + \cdots + \frac{\Delta^n}{n!}\right) 0^n,$$

the notation being that of finite differences, so that

$$\epsilon_n = \sum_{s=1}^{n} \frac{1}{(s-1)!}\left[\sum_{r=0}^{s-1} (-1)^r \binom{s-1}{r} (s-r)^{n-1}\right], \qquad n > 0.$$

[4] By Herschel's theorem in finite differences. The ϵ were given as an example in the paper mentioned in the introduction; the table up to ϵ_{20} here was computed by J. L. Bell by machine from Cayley's table of $\Delta^n 0^m/n!$ (Collected Papers, vol. 11, pp. 145–6.) The congruences for the 0's incidentally check Cayley's table. For the combinatorial meaning of the ϵ see Whitworth, Choice and Chance, p. 95.

As the ϵ are among the simplest ϕ for constant values of the variables, they are valuable for checking the general ϕ congruences. Accordingly we list the first 20, which will be sufficient.

n	ϵ_n	n	ϵ_n
1	1	20	51724158235372
2	2	19	5832742205057
3	5	18	682076806159
4	15	17	82864869804
5	52	16	10480142147
6	203	15	1382958545
7	877	14	190899322
8	4140	13	27644437
9	21147	12	4213597
10	115975	11	678570

In passing, the connection of the ϵ with certain other polynomials η, G, the second of which were introduced by Steffensen[5] in connection with Makeham's formula, may be noted.

The polynomials $\eta(x)$ are defined by

$$\eta_0(x) = 1, \qquad \eta_s(x) \equiv \sum_{n=1}^{s} x^n \frac{\Delta^n 0^s}{s!} \quad (s > 0),$$

so that $\eta(1) = \epsilon$. The generating identity is

$$\exp\,[x(e^t - 1)] = \exp\,[t\eta(x)];$$

whence, as a definition of G,

$$\exp[zt - x(e^t - 1)] \equiv \exp\,[tG(x, z)] = \exp\,[t(z + \eta(-x))].$$

Thus $G_n(x, z)$ is the Appell polynomial $(z + \eta(-x))^n$ in z;

$$\eta(x) = G(-x, 0); \qquad \epsilon = G(-1, 0);$$
$$G_n(x, z) = \phi_n(\beta), \qquad \beta_1 = z - x, \qquad \beta_s = -x(s > 0).$$

If in (4.3) t be replaced by ct, where c is constant, we see that

$$\phi_n(c\alpha_1, c^2\alpha_2, \cdots, c^n\alpha_n) = c^n\phi_n(\alpha_1, \alpha_2, \cdots, \alpha_n), \tag{4.8}$$

so that ϕ_n is isobaric in $\alpha_1, \cdots, \alpha_n$ of weight n. From (4.3) we get immediately the addition theorem

$$\phi_n(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_n + \beta_n) = [\phi(\alpha) + \phi(\beta)]^n. \tag{4.9}$$

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[5] J. F. Steffensen, *Some recent researches in the theory of statistics and actuarial science*, Cambridge Press, 1930, p. 24.

5. Write $D_{\alpha_j} \equiv D_j$. Then, from (4.3), we have at once

$$D_0\phi_n = 0, \qquad D_j\phi_n = \binom{n}{j}\phi_{n-j}\ (j > 0). \tag{5.1}$$

Thus $\phi_n(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is an Appell polynomial in α_1, and hence the $\phi_n(\alpha_1, \cdots, \alpha_n)$ may be considered as a generalization of the Appell polynomials.

6. The derivatives of the ϕ_n are of importance in the arithmetical properties. By repetitions of (5.1) we get

$$D_j^r\phi_n = \frac{n!}{(j!)^r\,(n - rj)!}\phi_{n-rj},\ j > 0, \qquad 0 < r \leqq [n/j], \tag{6.1}$$

$$D_j^r\phi_n = 0 \text{ if } r > [n/j].$$

From the way in which (6.1) is obtained it follows that the coefficient of ϕ_{n-rj} is a positive integer which is divisible by $\binom{n}{j}$. By (6.1) the degree of ϕ_n in α_j is $[n/j]$ $(j = 1, \cdots, n)$.

When, as in the general definition, the α are independent variables, a congruence involving them (or polynomials $\phi(\alpha)$) is interpreted to mean that the coefficients of the several power products of α's on both sides of the congruence are congruent with respect to the modulus. In particular, if all the coefficients are integers, the congruences hold for all sets of integer values of the α's.

As always henceforth, let p denote a rational prime. Then (5.1), (6.1) and the elementary congruence properties of binomial coefficients imply

$$D_j^r\phi_p \equiv 0 \bmod p, \qquad 0 < r, \qquad 0 < j < p. \tag{6.2}$$

Hence, by (4.4), for the same r, j,

$$D_j^r P(\alpha_1, \cdots, \alpha_{p-1}) \equiv 0 \bmod p, \tag{6.3}$$

where P is of degree $< p$. Consider a particular term of P, say

$$k\alpha_1^{\alpha_1} \cdots \alpha_j^{\alpha_j} \cdots \alpha^{\alpha_{p-1}};$$

k is an integer > 0. Since the α's are independent variables, it follows from (6.3) that

$$ka_j(a_j - 1) \cdots (a_j - r + 1) \equiv 0 \bmod p,$$

which holds for all integers $r > 0$. Take $r = 1$. Then $ka_j \equiv 0 \bmod p$, and therefore, since p is prime, at least one of $k \equiv 0 \bmod p$, $a_j \equiv 0 \bmod p$ holds. But $a_j \equiv 0 \bmod p$ is impossible, since the degree of P is $< p$. Hence $k \equiv 0 \bmod p$. Or, since $a_j, a_j - 1, \cdots, a_j - r + 1$ are all $< p$, and p is prime, it follows at once that $k \equiv 0 \bmod p$. This proves

$$\phi_p \equiv \alpha_1^p + \alpha_p \bmod p. \tag{6.4}$$

Again, from (4.1) we have

$$\phi^{p+r+1} = \alpha(\phi + \alpha)^{p+r},$$

and hence, by a general theorem established in a previous paper on anharmonic polynomials,[6]

$$\phi^{p+r+1} \equiv \alpha(\phi^p + \alpha^p)\,(\phi + \alpha)^r \bmod p, \quad 0 \leqq r < p,$$

which gives

$$\phi_{p+r+1} - \sum_{s=0}^{r} \binom{r}{s} \alpha_{s+1}\phi_{p+r+s} - \alpha^{p+1}(\phi + \alpha)^r \equiv 0 \bmod p.$$

In the last take r successively equal to $r-1, r-2, \cdots, 0$, and solve the resulting system of congruences (the system is obviously solvable) for ϕ_{p+r+1}. Then

$$(-1)^{r+1}\phi_{p+r+1} \tag{6.5}$$

is congruent mod p to

$$\begin{vmatrix}
\alpha^{p+1}(\phi+\alpha)^r & \alpha_1 & \binom{r}{1}\alpha_2 & \binom{r}{2}\alpha_3 & \binom{r}{3}\alpha_4 \cdots \binom{r}{r}\alpha_{r+1} \\
\alpha^{p+1}(\phi+\alpha)^{r-1} & -1 & \alpha_1 & \binom{r-1}{1}\alpha_2 & \binom{r-1}{2}\alpha_3 \cdots \binom{r-1}{r-1}\alpha_r \\
\alpha^{p+1}(\phi+\alpha)^{r-2} & 0 & -1 & \alpha_1 & \binom{r-2}{1}\alpha_2 \cdots \binom{r-2}{r-2}\alpha_{r-1} \\
\alpha^{p+1}(\phi+\alpha)^{r-3} & 0 & 0 & -1 & \alpha_1 \cdots \binom{r-3}{r-3}\alpha_{r-2} \\
\alpha^{p+1}(\phi+\alpha)^{r-4} & 0 & 0 & 0 & -1 \cdots \binom{r-4}{r-4}\alpha_{r-3} \\
\cdots & & \cdots & & \cdots \\
\alpha^{p+1}(\phi+\alpha)^0 & 0 & 0 & 0 & 0 \cdots \alpha_1 \\
\alpha_1{}^p + \alpha_p & 0 & 0 & 0 & 0 \cdots -1
\end{vmatrix}$$

which holds for $0 \leqq r < p$. The binomial coefficient $\binom{0}{0}$ in this is by convention 1. For example, taking $r = 0, 1, 2$ we get

$$\begin{aligned}
\phi_{p+1} &\equiv \alpha_1{}^{p+1} + \alpha_1\alpha_p + \alpha_{p+1} \bmod p, \\
\phi_{p+2} &\equiv (\alpha_1^2 + \alpha_2)\alpha_p + 2\alpha_1\alpha_{p+1} + \alpha_{p+2} + \alpha_1{}^{p+2} + \alpha_2\alpha_1{}^p \bmod p, \\
\phi_{p+3} &\equiv 3(\alpha_1^2 + \alpha_2)\alpha_{p+1} + 3\alpha_1\alpha_{p+2} + \alpha_{p+3} + (\alpha_1^3 + 3\alpha_1\alpha_2 + \alpha_3)\alpha_p \\
&\quad + \alpha_1{}^{p+3} + 3\alpha_2\alpha_1{}^{p+1} + \alpha_3\alpha_1{}^p \bmod p, \qquad p > 2,
\end{aligned} \tag{6.6}$$

and these may be verified for the ϵ by setting each $\alpha = 1$. With (6.4) we have thus, mod p,

$$\epsilon_p \equiv 2, \qquad \epsilon_{p+1} \equiv 3, \qquad \epsilon_{p+2} \equiv 7; \qquad \epsilon_{p+3} \equiv 20 \ (p > 2).$$

The condition $p > 2$ in the last happens to be superfluous for the ϵ, but this does not follow from the proof of the general congruence. These congruences are checked by the values in the table in §4.

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[6] *Trans. American Math. Soc.*, vol. 34, 1922, p. 109.

The congruence (6.5) can be considerably generalized by applying theorems proved in a previous paper[7] on residues of certain types of binomial coefficients. For example, it has shown that

$$\binom{2p}{h} \equiv 2\binom{p}{h} \bmod p^2, \qquad 0 < h < p;$$

$$\binom{2p}{h} \equiv 2\binom{p}{h-p} \bmod p^2, \qquad p < h < 2p.$$

Hence, proceeding as above, we get

$$(6.6)\quad \phi_{2p+1} \equiv \alpha_{2p+1} + \binom{2p}{p}\alpha_{p+1}\phi_p + \alpha_1\phi_{2p} + 2\sum_{h=1}^{p-1}\binom{p}{h}(\alpha_{h+1}\phi_{2p-h} + \alpha_{p+h+1}\phi_{p-h}) \bmod p^2.$$

Similar results may be found for moduli p^a, $p^a q^b$, $p^a q^b r^c$, $\cdots$, where $p, q, r, \cdots$ are distinct primes, and $a, b, c, \cdots$ are any integers > 0, by means of the theorems cited.

Having derived any congruence of the type (6.5) for a particular r, we can obtain from it congruences for $r - s (s > 0)$ by differentiating the given congruence and applying (6.1). For this it is more convenient to rewrite (6.1) as

$$D_j^s\phi_n = \binom{n}{j}\binom{n-j}{j}\binom{n-2j}{j}\cdots\binom{n-(s-1)j}{j}\phi_{n-sj}.$$

For, (6.5) is equivalent to an identity in the α of the form

$$\phi_{p+r+1} \equiv P(\alpha_{p+r+1}, \alpha_{p+r}, \cdots, \alpha_p, \alpha_1, \alpha_2, \cdots, \alpha_{r+1}) + pQ(\alpha_1, \alpha_2, \cdots, \alpha_{p+r+1}),$$

where P, Q are polynomials, with integer coefficients; P is $(-1)^{r+1}$ times the determinant in (6.5). Hence

$$(6.7)\qquad D_j^s\phi_{p+r+1} \equiv D_j^s P \bmod p.$$

The coefficients on both sides of (6.7) are integers; the left of (6.7) is $\phi_{p+r+1-sj}$ times the integer

$$\binom{p+r+1}{j}\binom{p+r+1-j}{j}\binom{p+r+1-2j}{j}\cdots\binom{p+r+1-(s-1)j}{j}.$$

If the left of (6.7) is not to vanish identically, s, j must be chosen such that $p + r + 1 - sj \geqq 0$; the maximum value of j is $p + r + 1$. Subject to these

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[7] *Journal of the London Math. Soc.*, vol. 5, 1930, pp. 253-258.

restrictions the above product of binomial coefficients is reduced modulo p by means of

$$\binom{m}{n} \equiv \binom{m_1}{n_1}\binom{a}{b} \bmod p,$$

$$m > 0,\ n > 0,\ m = m_1 p + a, \qquad n = n_1 p + b,$$

$$0 \leqq a, \qquad b < p, \qquad \binom{a}{b} = 0 \text{ if } b > a.$$

The result of this reduction is an integer, say c. Finally then, from (6.7) we have

(6.8)
$$c\phi_{p+r+1-sj} \equiv D_j^s P \bmod p,$$
$$j > 0, \qquad p + r + 1 \geqq sj, \qquad s \geqq 0.$$

7. The polynomials Y_n next considered are a generalization of the ξ_n. Write (see (4.3))

(7.1)
$$y \equiv e^{2t} - \alpha_0 = \alpha_1 t + \alpha_2 t^2/2! + \cdots + \alpha_n t^n/n! + \cdots,$$

and define the Y_n by

(7.2)
$$e^{-y} D_t^n e^y \equiv Y_n, \qquad n = 0, 1, \cdots; \qquad Y_0 = 1.$$

Obviously, with $Y_s \equiv D_t^s y$ we have

(7.3)
$$Y_n \equiv Y_n(y_1, \cdots, y_n),$$

and if $\alpha_r = r!\, x$, $\alpha_n = 0$, $n \neq r$, then $Y_n = \xi_n(x, t; r)$.

Operating on (7.2) with D_t we get

(7.4)
$$Y_{n+1} = (y_1 + D_t) Y_n,$$

which corresponds to (1.2). For checks on subsequent formulas we write down from (7.4)

$$Y_1 = y_1, \qquad Y_2 = y_1^2 + y_2, \qquad Y_3 = y_1^3 + 3y_1y_2 + y_3,$$
$$Y_4 = y_1^4 + 6y_1^2y_2 + 4y_1y_2 + 3y_2^2 + y_4,$$

which may be compared with the corresponding ϕ_n in §4. In fact we have generally

(7.5)
$$Y_n = \phi_n(y_1, \cdots, y_n).$$

To prove (7.5) we observe from (4.2), (6.1) that

(7.6)
$$\phi_{n+1} = \left(\alpha_1 + \sum_{s=1}^{n} \alpha_{s+1} \frac{\partial}{\partial \alpha_s}\right)\phi_n.$$

Let $k y_1^{a_1} \cdots y_n^{a_n}$ be a particular term of Y_n. Then, since $D_t y_s = y_{s+1}$,

we have

$$D_t(ky_1^{a_1}\cdots y_n^{a_n}) = \left(y_2\frac{\partial}{\partial y_1}+\cdots+y_{n+1}\frac{\partial}{\partial y_n}\right)(ky_1^{a_1}\cdots y_n^{a_n}),$$

and therefore by (7.4),

$$Y_{n+1} = \left(y_1+\sum_{s=1}^{n} y_{s+1}\frac{\partial}{\partial y_s}\right)Y_n. \tag{7.7}$$

These functions are connected with the determinant in (6.5). For, by the definition of y_s we have

$$y_s = \alpha^s e^{\alpha t} = \sum_{n=0}^{\infty}\alpha_{n+s}t^n/n!\ (s>0),$$

and if we define $\alpha^{(s)}$ by the MacLaurin expansion of Y_s in powers of t thus,

$$Y_s = \exp(t\alpha^{(s)}) \equiv \sum_{n=0}^{\infty}\alpha_n^{(s)}t^n/n!, \tag{7.8}$$

we see from (7.1), (4.3), (7.8) that

$$Y_s e^y = \phi^s e^{t\phi} = \exp[t(\alpha^{(s)}+\phi)],$$

and hence

$$\phi_{n+s} = (\phi+\alpha^{(s)})^n. \tag{7.9}$$

In (7.9) take $n=p$, $s=r+1$, as in (6.5). Then, referring to (6.4), we have

$$(-1)^{r+1}\phi_{p+r+1} \equiv \alpha_1^p+\alpha_p+\alpha_p^{(r+1)} \bmod p. \tag{7.10}$$

The symbolic recurrence for Y,

$$Y^{n+1} = y(Y+y)^n, \qquad Y_0 = 1, \qquad y^n \equiv y_n, \tag{7.11}$$

follows from (7.6), (7.7) compared with (4.1). From the last we deduce the recurrence for the $\alpha^{(s)}$. For,

$$Y_{n+1} = \sum_{s=0}^{n}\binom{n}{s}Y_{n-s}y_{s+1},\ Y_{n+s} = \exp(t\alpha^{(n-s)}), \qquad y_{s+1} = \alpha^{s+1}e^{t\alpha};$$

hence, on substituting in the first for Y_{n-s}, y_{s+1} and comparing coefficients of t^m, we have

$$\alpha_m^{(n+1)} = \sum_{s=0}^{n}\binom{n}{s}\sum_{r=0}^{m}\binom{m}{r}\alpha_{m-r+s+1}\alpha_r^{(n-s)}, \tag{7.12}$$

which may be written

$$\alpha_m^{(n+1)} = \sum_{s=0}^{n} \binom{n}{s} \alpha^{s+1} (\alpha + \alpha^{(n-s)})^m. \tag{7.13}$$

In the congruences for the ϕ we are interested only in the residues mod p of the $\alpha^{(s)}$, as in (7.10). From (7.13) we have

$$\alpha_m^{(p+1)} \equiv \alpha(\alpha + \alpha^{(p)})^m + \alpha_{m+p+1} \bmod p, \tag{7.14}$$

or, in non-symbolic form,

$$\alpha_m^{(p+1)} \equiv \sum_{s=0}^{m} \binom{m}{s} \alpha_{m-s+1} \alpha_s^{(p)} + \alpha_{m+p+1} \bmod p. \tag{7.15}$$

Considering Y_n as a function $Y_n(t)$ of t we have

$$e^{t\phi} = e^y = \sum_{n=0}^{\infty} \frac{t^n}{n!} [D_t^n(e^y)]_{t=0};$$

hence

$$\phi_n \equiv \phi_n(\alpha) = Y_n(0) = \alpha_0^{(n)};\ Y_n(0) \equiv [Y_n(t)]_{t=0}. \tag{7.16}$$

8. By the remark after (7.3) and §2 it follows that *the Y_n satisfy no differential equation of the type*

$$R_0 D_t^m Y_n + R_1 D_t^{m-1} Y_n + \cdots + R_m = R_{m+1}$$

of constant order m, where the R's are polynomials in α's and t. Otherwise we should have a contradiction with §2 for suitably chosen α's.

9. For properly restricted α, the Y_n give rise to an orthogonal set in $-\infty \leqq t \leqq +\infty$. The restrictions are $\alpha_{2n} > 0$, $\alpha_{2n-1} = 0 (n = 1, 2, \cdots)$. For these α, write $Y_n \equiv Y_n^1$, and let $y \equiv y(t)$, for the same α, have a radius of convergence > 0. Write $Z_n \equiv e^{-y/2} Y_n^1$. Then, as in §3, it may be shown that

$$\int_{-\infty}^{\infty} Z_m Z_n \, dt = 0, \qquad m \neq n.$$

10. Returning to the ϕ, we shall give some miscellaneous properties which follow readily from what has been given. Let 0_s denote the sequence $0, \cdots, 0$ of s zeros. Then

$$k!\, \phi_{km}\left(0_{m-1}, \frac{m!}{1!}\alpha_1, 0_{m-1}, \frac{(2m)!}{2!}\alpha_2, 0_{m-1}, \cdots, 0_{m-1}, \frac{(km)!}{k!}\alpha_k\right) = (km)!\, \phi_k(\alpha_1, \cdots, \alpha_k), \tag{10.1}$$

where 0_0 indicates that the symbol 0_s, $s = 0$, is to be suppressed. This follows from (y as in §7)

$$t^h \exp y(t^m) = \sum_{n=0}^{\infty} \phi_n(\alpha_1, \cdots, \alpha_n) t^{mn+h}/n! .$$

The integrals are found from (5.1). Let $\phi_n^{(j)}$ denote the result of putting $\alpha_j = 0$ in ϕ_n. Then

$$\phi_n^{(j)} = \sum_{s=0} \frac{(-1)^s}{s!} \alpha_j^s D_j^s \phi_n, \tag{10.2}$$

as we see on multiplying exp y by exp $(-\alpha_j t^j/j\,!)$. From (5.1),

$$\binom{n+j}{j} \int_0^{\alpha_j} \phi_n \, d\alpha_j = \phi_{n+j} - \phi_{n+j}^{(j)},$$

which may be written, by (10.2), (6.1),

$$\int_0^{\alpha_j} \phi_n \, d\alpha_j = n! \sum_{s=0}^{[n/j]} \frac{(-1)^s}{(j!)^s (s+1)!\,(n-sj)!} \alpha_j^{s+1} \phi_{n-sj}. \tag{10.3}$$

The differences of the ϕ are found by applying (6.1) to the following formula from finite differences,

$$\Delta_j^r = D_j^r + \frac{\Delta^r 0^{r+1}}{(r+1)!} D_j^{r+2} + \frac{\Delta^r 0^{r+2}}{(r+2)!} D_j^{r+2} + \cdots,$$

where Δ_j^r indicates the rth difference with respect to α_j. Thus

$$\Delta_j^r \phi_n = \sum_{s=0}^{[(n-rj)/s]} \frac{\Delta^r 0^{r+s}}{(r+s)!} \frac{n!}{(j!)^{r+s}(n-(r+s)j)!} \phi_{n-(r+s)j}; \tag{10.4}$$

the coefficients are positive integers.

Combined with previous results the last shows that, in an obvious sense, *the set of all ϕ is closed under operations of differentiation, integration, and finite differencing.*

Any polynomial (or power series, if convergent) in $\alpha_1, \cdots, \alpha_r$ can be expressed as a polynomial (or power series) in suitably chosen ϕ's. For we have

$$\alpha_s^n = \frac{(-1)^{sn}(s!)^n n!}{(sn)!} \phi_{sn}^{(s)}(k_s\alpha_1 - \phi, \alpha_2, \alpha_3, \cdots, \alpha_{sn}), \tag{10.5}$$

where $k_1 = 1$, $k_s = 0$, $s > 1$. The proof is from the identity exp y = exp $(t\phi)$, which gives an exponential expression for exp $(\alpha_r t^r/r!)$. By the formula preceding (10.3), the last gives the curious result

$$\alpha_s^n = \frac{(-1)^{sn}(s!)^n n!}{(sn)!} \left[\phi_{sn} - \binom{sn}{s} \int_0^{\alpha_s} \phi_{sn-s}(k_s\alpha_1 - \phi, \alpha_2, \cdots, \alpha_{sn}) d\alpha_s \right].$$

When y(as in §7) satisfies a differential equation of the type

$$(D_t^m + A_1 D_t^{m-1} + \cdots + A_{m-1}D_t + A_m)y = A_{m+1},$$

or, what is the same,

$$y_m + A_1 y_{m-1} + \cdots + A_{m-1}y_1 + A_m y = A_{m+1}, \tag{10.6}$$

where the A's are functions of t, some or all of which may be constants, possessing derivatives of all orders, the ϕ_n will be connected by relations of the form

$$R(\phi_1, \cdots, \phi_{m+s+1}) = 0, \qquad s \geqq m + 2, \tag{10.7}$$

where R is a polynomial of degree $m + 2$. The ϕ, Y are connected by (7.16). Hence we find a relation for the Y from which (10.7) follows by setting $t = 0$. From (7.11) we have

$$Y_{m+n+1} = y_1 Y_{m+n} + \binom{m+n}{1} y_2 Y_{m+n-1} + \cdots + y_{m+n+1}Y_0, \tag{10.8}$$

and $Y_0 = 1$; from (10.6), by successive operations with D_t and reductions after each by means of (10.6), y_{m+s} is expressed in a linear function of $y, y_1, \cdots, y_{m-1}$. Thus (10.8) is equivalent to

$$Y_{m+n+1} = \lambda_0(n)y + \lambda_1(n)y_1 + \cdots + \lambda_{m-1}(n)y_{m-1}, \tag{10.9}$$

where the λ's do not contain $y, y_1, \cdots, y_{m-1}$, and are linear functions of $Y_1, Y_2, \cdots, Y_{m+n}$. Give to n in (10.9) $m + 2$ distinct integer values > 0, of which the greatest is s, and eliminate $y, y_1, \cdots, y_{m-1}$. Setting $t = 0$ in the result gives (10.7).

It is obvious that ϕ_n, in the general case of unrestricted α or y, can satisfy no differential or difference equation of constant finite order, for all n, whose coefficients involve only a finite number of the α, since $\phi_n \equiv \phi_n(\alpha_1, \cdots, \alpha_n)$.

The general discussion has included the case when certain of the α's are replaced by zeros; in particular (10.7) holds when $y \equiv y(t)$ is a polynomial of degree m.

III. The Polynomials x.

11. The Appell polynomial in x (ordinary variable) to the *base* β (umbra) of rank n is $(x + \beta)^n (n = 0, 1, \cdots)$, namely $\sum_{s=0}^{n} \binom{n}{s} \beta_s x^{n-s}$, and this polynomial is generated by $\exp(hx) \times \exp(h\beta)$, where h is a parameter, since

$$\exp(hx)\exp(h\beta) = \exp[h(x + \beta)].$$

The obvious property $D_x(x + \beta)^n = n(x + \beta)^{n-1}$ is that which makes possible the discussion of the differential equation satisfied by $(x + \beta)^n$ when it is given that $\exp(h\beta)$ satisfies a differential equation of prescribed type (with respect to h).

In line with the polynomials ξ, ϕ, we now generalize the Appell polynomials to the polynomials ψ, defined by

$$\psi_n \equiv \psi_n(x, \beta; r),\ \psi_{-n-1} = 0,\ n = 0, 1, \cdots; \quad \exp(h^r x) \times \exp(h\beta) \equiv \exp(h\psi), \tag{11.1}$$

where r is a constant integer > 0. Hence $\psi_n(x, \beta; 1)$ is the general Appell polynomial of rank n to the base β. Obviously if $r > 1$, and β is an arbitrary sequence, $D_x\psi_n \neq n\psi_{n-1}$, so that the differential equation satisfied by ψ_n when the equation satisfied by $\exp(h\beta)$ is given, cannot be found by the same process as for Appell's polynomials. We shall see in fact that the equation for $\exp(h\beta)$ must be of a certain restricted type if the equation for ψ_n is to be of constant order (independent of n). The restriction degenerates to no restriction when $r = 1$.

From (11.1), by D_t, D_x^s we have

$$\psi_{m+1} = \beta\psi^m + \frac{m!\,rx}{(m-r+1)!}\psi_{m-r+1}\ (m \geqq 0), \qquad \psi_0 = \beta_0; \tag{11.2}$$

$$D_x^s\psi_m = \frac{m!}{(m-rs)!}\psi_{m-rs}\ (0 \leqq s \leqq [m/r]), \tag{11.3}$$

$$D_x^s\psi_m = 0\ (s > [m/r]),$$

and by D_β^s,

$$D_\beta^s\psi_m = m(m-1)\cdots(m-s+1)\psi_{m-s}. \tag{11.4}$$

Directly from (11.1) the explicit form is

$$\psi_m = m!\sum_{s=0}^{[m/r]} \frac{\beta_{m-sr}x^s}{(m-sr)!\,s!}; \tag{11.5}$$

the coefficient of $\beta_{m-sr}x^s$ is a non-negative integer, and there is the addition theorem

$$\psi_m(x+y, \beta; r) = [\psi(x, \beta; r) + \psi(y, \beta; r)]^m. \tag{11.6}$$

Writing (11.5) in the form

$$\psi_m = \sum_{s=0}^{[m/r]} (sr-s)!\binom{sr}{r}\binom{m}{sr}\beta_{m-sr}x^s,$$

we see that (p prime)

$$\psi_p(x, \beta; r) \equiv \beta_p \bmod p, \qquad r \not\equiv p. \tag{11.7}$$

The addition theorem may be written

$$\psi_m(x+y, \beta; r) = \psi_m(\psi(x, \beta; r), y; r), \tag{11.8}$$

and x, y are interchangeable on the right. This follows by inspection from (11.1).

12. We now examine the possible differential equations of order independent of n satisfied by ψ_n when $\exp(h\beta)$ satisfies a linear differential equation (with respect to h) with polynomial coefficients in h. For this we introduce the intermediary functions (not polynomials)

$$\psi_n^{(s)} \equiv \psi_n^{(s)}(x, \beta; r), \qquad \psi_{-n-1}^{(s)} \equiv 0 \ (n = 0, 1, \cdots ; s > 0),$$

defined by

$$\exp(xh^r) D_h^s \exp(h\beta) \equiv \exp(h\psi^{(s)}). \tag{12.1}$$

Take D_h^s of (11.1) and refer to (1.1). Then

$$\psi^s \exp(h\psi) = \sum_{j=0}^{s} \binom{s}{j} \exp(h\psi^{(s-j)}),$$

$$\xi_j \equiv \xi_j(x, h; r), \ \psi_n^{(s-j)} \equiv \psi_n^{(s-j)}(x, \beta; r), \ \psi_n \equiv \psi_n(x, \beta; r).$$

Hence, by (1.5),

$$\exp(h\psi^{(s)}) = \psi^s \exp(h\psi) - \sum_{j=0}^{s} \binom{s}{j} \exp(h\psi^{(s-j)}) \sum_{i=0} c_i(j) x^{j-i} h^{(j-i)r-j}, \tag{12.2}$$

from which $\psi_m^{(s)}$ can be calculated. It will be sufficient to show that

$$\psi_m^{(s)} = \sum_{a=0}^{s} A_a^{(s)} D_x^a \psi_{m+s}, \qquad A_0^{(s)} = 1, \tag{12.3}$$

where the point to be noted is that only derivatives of the *single* ψ function ψ_{m+s} appear on the right. The precise form of the A's is immaterial; actually we shall prove that

$$A_0^{(s+1)} = A_0^{(s)} - \frac{rx}{m+1} D_x A_0^{(s)}, \quad A_{s+1}^{(s+1)} = -\frac{rx}{m+1} A_s^{(s)}, \tag{12.4}$$

$$A_a^{(s+1)} = A_a^{(s)} - \frac{rx}{m+1}(D_x A_a^{(s)} + A_{a-1}^{(s)}) \qquad (a = 1, \cdots, s),$$

whence the A's can be found explicitly if desired. The proof will be from (12.3) by mathematical induction on s.

Taking D_h of (12.1), and comparing coefficients of h^m in the result, we get

$$r! \binom{m}{r-1} x \psi_{m-r+1}^{(s)} + \psi_m^{(s+1)} = \psi_{m+1}^{(s)}; \tag{12.5}$$

and D_x of (12.1) gives

$$D_x \psi_m^{(s)} = r! \binom{m}{r} \psi_{m-r}^{(s)}. \tag{12.6}$$

From (12.6), (12.5) we have

$$\frac{rx}{m+1}D_x\psi_{m+1}^{(s)} + \psi_m^{(s+1)} = \psi_{m+1}^{(s)}. \tag{12.7}$$

In (12.3) replace m by $m+1$ and apply (12.7); then

$$\psi_{m+1}^{(s+1)} + \frac{rx}{m+1}D_x\psi_{m+1}^{(s)} = \sum_{a=0}^{s} A_a^{(s)}D_x\psi_{m+1+s}. \tag{12.8}$$

Now (merely to simplify the writing) define

$$P_a^{(s)} \equiv A_a^{(s)} - \frac{rx}{m+1}D_xA_a^{(s)}, \quad -Q_a^{s)} \equiv \frac{rx}{m+1}A_a^{(s)}.$$

To the second term of (12.8) apply (12.3) with m replaced by $m+1$ in (12.3). Then

$$\psi_m^{(s+1)} = \sum_{a=0}^{s}[P_a^{(s)}D_x^a\psi_{m+1+s} + Q_a^{(s)}D_x^{a+1}\psi_{m+1+s}], \tag{12.9}$$

and therefore, with the $A^{(s+1)}$ defined in (12.4) we have completed the induction on s (the case $s=1$ obviously is true).

From the foregoing we can construct the required differential equations. Let c be independent of t, and let k be an arbitrary non-negative integer. Then, from (12.1), (12.3),

$$ch^k\exp(xt^r)D_h^s(\exp(h\beta)) = c\sum_{n=0}^{\infty}\frac{h^{n+k}}{n!}\sum_{a=0}^{s}A_a^{(s)}D_x^a\psi_{n+s}, \tag{12.10}$$

the second member of which is equal to

$$c\sum_{m=k}^{\infty}\frac{h^m}{(m-k)!}\sum_{a=0}^{s}A_a^{(s)}D_x^a\psi_{m+s-k}.$$

For our purpose, as stated at the beginning of this section, it is now required to choose s, k in the last so that all derivatives D_x^a are taken with respect to the *same* function ψ. From (11.3) we have

$$D_x^a\,\psi_{m+s-k} = (ra)!\binom{m+s-k}{ra}\psi_{m+s-k-ra}; \tag{12.11}$$

the restriction on a need not be included, since $\psi_{-j}=0$, $j>0$. Hence we choose $s-k \equiv q \bmod r$; say $s-k=q-ur$. Application of (11.3) to (12.11) now gives

$$D_x^a\,\psi_{m+s-k} = KD_x^{a+u}\psi_{m+q},$$

$$K \equiv (ra)!\binom{m+q-ur}{ra}/(r(a+u))!\binom{m+q}{r(a+u)}; \tag{12.12}$$

and therefore, from (12.10),

$$\exp(xt^r)\,[ch^kD_h^s(\exp(h\beta))] = c\sum_{m=k}^{\infty}\frac{h^m}{(m-k)!}\sum_{a=0}^{s}A_a^{(s)}KD_x^{a+u}\psi_{m+q},$$

where $s = q - k - ur$.

It follows that *if* $\exp(h\beta)$ *satisfies the differential equation*

$$\sum_{c,k,u} ch^kD_h^{q-k-ur}(\exp(h\beta)) = 0,$$

then the polynomials ψ_{m+q} *satisfy the differential equation*

$$\sum_{c,k,u}\frac{c}{(m-k)!}\sum_{a=0}^{q-k-ur}A_a^{(q-k-ur)}KD_x^{a+u}\psi_{m+q} = 0,$$

whose order is independent of m, *where the constants* K *are as in* (12.12), *and the* A*'s are given by* (12.4).

The derivatives in the given equation for $\exp(h\beta)$ are of orders in arithmetical progression with common difference r; when the ψ's are Appell polynomials ($r = 1$), the implied restriction disappears.

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SUR LES FONCTIONS JOUISSANT DE LA PROPRIÉTÉ DE BAIRE DE FONCTIONS CONTINUES

PAR WACLAW SIERPINSKI

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On dit qu'une fonction d'une variable réelle *jouit de la propriété de Baire*, si elle est continue sur tout ensemble parfait P quand on néglige les ensembles de première catégorie de Baire par rapport à l'ensemble P.

En 1905 M. H. Lebesgue, après avoir démontré le théorème de Baire, d'après lequel toute fonction représentable analytiquement jouit de la propriété de Baire, posa la question si le théorème inverse a encore lieu.[1] Cette question a été résolue négativement par M. N. Lusin en 1914 à l'aide de l'hypothèse du continu[2] et en 1917 sans utiliser cette hypothèse.[3]

Il existe une analogie, et même une certaine dualité, entre les fonctions jouissant de la propriété de Baire et les fonctions mesurables. P. e. la somme et le produit d'un nombre fini de fonctions jouissant de la propriété de Baire (resp mesurables) est une fonction jouissant de la propriété de Baire (resp. mesurable); la propriété de Baire (comme la propriété d'être une fonction mesurable) subsiste au passage à la limite (d'une suite convergente de fonctions qui jouissent de cette propriété).[4]

Cependant les démonstrations pour les fonctions mesurables et pour les fonctions jouissant de la propriété de Baire sont différentes, et pour les fonctions jouissant de la propriété de Baire souvent beaucoup plus difficiles que pour les fonctions mesurables. On démontre p. e. sans peine qu'il existe une fonction mesurable qui ne jouit pas de la propriété de Baire, mais les démonstrations connues d'existence de fonctions jouissant de la propriété de Baire et non mesurables utilisent l'hypothèse du continu.[5]

Pareillement, on démontre sans peine qu'une fonction mesurable d'une fonction continue peut être non mesurable. Le problème si une fonction jouissant de la propriété de Baire d'une fonction continue jouit de la propriété de Baire restait ouvert.[6] Le but de cette Note est de prouver, en admettant l'hypothèse du continu, que la solution de ce problème est négative. Nous démontrerons notamment ce

THÉORÈME.[7] *Si la puissance du continu est aleph-un, il existe une fonction*

[1] Journal de Mathématiques 1905, p. 188.
[2] C. R. Paris, t. 158, p. 1259.
[3] Fundamenta Mathematicae t. II, p. 157.
[4] Fundamenta Mathematicae t. I, p. 163.
[5] Voir N. Lusin, Fund. Math. t. IX, p. 116 et S. Saks, Fund. Math. t. XI, p. 217.
[6] Cf. Fund. Math. t. XX, p. 286 (Problème 59).
[7] J'ai signalé ce théorème dans les C. R. Paris, t. 197, p. 1716–1717.

d'une variable réelle qui ne jouit pas de la propriété de Baire et qui est une fonction jouissant de la propriété de Baire d'une fonction continue.

Démonstration. M. N. Lusin a démontré que si $2^{\aleph_0} = \aleph_1$, il existe un ensemble non dénombrable de nombres irrationnels de l'intervalle (0, 1), L, qui est au plus dénombrable sur tout ensemble linéaire non dense[8] et qu'il existe une fonction $g(x)$ définie et continue sur l'ensemble N de tous les nombres irrationnels de l'intervalle (0, 1) à valeurs distinctes sur N et appartenant à l'intervalle (0, 1), et qui transforme l'ensemble L en un ensemble $G = g(L)$ qui est toujours de première catégorie (c'est-à-dire de I^re^ catégorie sur tout ensemble parfait).[9] On voit sans peine qu'on peut encore supposer que $G \subset N$.

Soit

$$x = \varphi(t) \qquad y = \psi(t) \qquad (0 \leq t \leq 1) \tag{1}$$

une courbe continue remplissant le carré $Q(0 \leq x \leq 1, 0 \leq y \leq 1)$. On peut, comme on sait, définir cette courbe de telle façon que les points du carré Q, dont toutes les deux coordonnées sont irrationnelles, soient points simples de la courbe (1), c'est-à-dire qu'il existe pour tout système (x_0, y_0) de deux nombres irrationnels x_0 et y_0 de l'intervalle (0, 1) un seul nombre réel t de l'intervalle (0, 1), tel que $x_0 = \varphi(t)$ et $y_0 = \psi(t)$ (Telles sont p. e. les courbes continues remplissants le carré définies par G. Peano ou par M. D. Hilbert).

Posons encore $\psi(x) = \psi(0)$ pour $x < 0$ et $\psi(x) = \psi(1)$ pour $x > 1$: la fonction $\psi(x)$ sera ainsi définie et continue pour tous les x réels.

Désignons maintenant par E l'ensemble de tous les nombres réels t de l'intervalle (0, 1), tels que

$$\varphi(t) \in L \quad \text{et} \quad \psi(t) = g(\varphi(t)). \tag{2}$$

Je dis que l'ensemble E est homéomorphe à l'ensemble L, et que la fonction $\varphi(t)$ établit une homéomorphie entre les points t de E et les points $x = \varphi(t)$ de L.

Soit, en effet, t un nombre de E: d'après la définition de l'ensemble E on a donc les formules (2): $x = \varphi(t)$ est donc un élément de l'ensemble L. On a ainsi $\varphi(E) \subset L$. D'autre part, soit x un élément de l'ensemble L. D'après $L \subset N$ on a donc $x \in N$ et $g(x)$ est un nombre bien déterminé de l'intervalle (0, 1). La courbe (1) remplissant le carré Q, il existe donc un nombre réel t (de l'intervalle (0, 1)), tel que

$$\varphi(t) = x \quad \text{et} \quad \psi(t) = g(x):$$

d'après $x \in L$, nous concluons donc (vu la définition de l'ensemble E) que $t \in E$. On a donc $x = \varphi(t) \in \varphi(E)$. Nous avons ainsi démontré que $L \subset \varphi(E)$ et, comme nous avons trouvé plus haut $\varphi(E) \subset L$, nous avons: $L = \varphi(E)$. Or,

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[8] C. R. Paris t. 158, p. 1259.

[9] Fundamenta Mathematicae t. XXI, p. 119–122.

je dis que la fonction $\varphi(t)$ est à valeurs distinctes sur E. Soient t_1 et $t_2 \neq t_1$ deux nombres de E et supposons que

$$\varphi(t_1) = \varphi(t_2). \tag{3}$$

D'après $t_1 \in E$, $t_2 \in E$ et d'après la définition de l'ensemble E, on a

$$\varphi(t_1) \in L, \varphi(t_2) \in L \tag{4}$$

et

$$\psi(t_1) = g(\varphi(t_1)) \text{ et } \psi(t_2) = g(\varphi(t_2)). \tag{5}$$

De (4) et de $L \subset N$ résulte que $\varphi(t_1) \in N$ et $\varphi(t_2) \in N$ et de (4), (5) et de $G \subset N$ résulte que $\psi(t_1) = g(\varphi(t_1)) \in g(L) = G \subset N$, donc $\psi(t_1) \in N$ et pareillement $\psi(t_2) \in N$ Or, d'après (3) et (5) on a

$$\varphi(t_1) = \varphi(t_2) \text{ et } \psi(t_1) = \psi(t_2), \tag{6}$$

les nombres $\varphi(t_1)$, $\varphi(t_2)$, $\psi(t_1)$ et $\psi(t_2)$ étant tous les quatre irrationnels (comme appartenant à N), il résulte de (6) et de la propriété de la courbe(1) que $t_1 = t_2$, contrairement à l'hypothèse.

Nous avons ainsi démontré que les formules

$$t_1 \in E, t_2 \in E \text{ et } t_1 \neq t_2$$

entrainent l'inégalité

$$\varphi(t_1) \neq \varphi(t_2).$$

La fonction $x = \varphi(t)$ établit donc une correspondance biunivoque et continue entre les points de E et les points de L. Désignons pour tout nombre x de L par $\vartheta(x)$ le nombre (unique) t de E tel que $x = \varphi(t)$. Pour prouver que la fonction $\varphi(t)$ établit une homéomorphie entre les ensembles E et L il nous reste donc a démontrer que la fonction $\vartheta(x)$ est continue dans l'ensemble L.

Soit donc x_0 un point de l'ensemble L et soit $x_n (n = 1, 2, 3, \cdots)$ une suite infinie de points de L, telle que

$$\lim_{n=\infty} x_n = x_0. \tag{7}$$

Admettons que la formule

$$\lim_{n=\infty} \vartheta(x_n) = \vartheta(x_0) \tag{8}$$

ne soit pas vraie. Il existe dans ce cas un nombre positif δ et une suite infinie croissante d'indices $K_n (n = 1, 2, 3. \cdots)$, telle que

$$| \vartheta(x_{K_n}) - \vartheta(x_0) | \geq \delta \text{ pour } n = 1, 2, 3, \cdots . \tag{9}$$

La suite infinie $\vartheta(x_{K_n})$ $(n = 1, 2, 3, \cdots)$, en tant que formée de nombres de l'intervalle (0, 1), étant bornée, il existe une suite infinie d'indices croissants $j_n (n = 1, 2, 3, \cdots)$, telle que la limite

$$u_0 = \lim_{n=\infty} \vartheta(x_{K_{j_n}}) \tag{10}$$

existe et, d'après (9) on a $| u_0 - \vartheta(x_0) | \geq \delta$ donc

$$u_0 \neq \vartheta(x_0). \tag{11}$$

De la définition de la fonction ϑ et de $x_n \in L$ pour $n = 1, 2, 3, \cdots$, résulte que

$$\varphi(\vartheta(x_n)) = x_n \text{ pour } n = 1, 2, 3, \cdots, \tag{12}$$

donc, d'après (7):

$$\lim_{n=\infty} \varphi(\vartheta(x_n)) = x_0. \tag{13}$$

Or, posons, pour abréger:

$$x'_n = x_{K_{j_n}} \text{ pour } n = 1, 2, 3, \cdots. \tag{14}$$

La fonction φ étant continue dans l'intervalle (0, 1) on a, d'après (10) et (14):

$$\lim_{n=\infty} \varphi(\vartheta(x'_n)) = \varphi(u_0),$$

donc, d'après (13) et (14):

$$\varphi(u_0) = x_0. \tag{15}$$

Or, de $x_0 \in L$ résulte que $\varphi(\vartheta(x_0)) = x_0$: la formule (15) donne donc

$$\varphi(u_0) = \varphi(\vartheta(x_0)). \tag{16}$$

Or, la fonction ψ étant continue dans (0, 1), on a, d'après (10) et (14):

$$\psi(u_0) = \lim_{n=\infty} \psi(\vartheta(x'_n)). \tag{17}$$

D'après $x_n \in L$ pour $n = 1, 2, \cdots$ et (14), on a $x'_n \in L$ pour $n = 1, 2, 3, \cdots$, donc, d'après (7) et $x_0 \in L \subset N$, la fonction g étant continue dans N:

$$\lim_{n=\infty} g(x'_n) = g(x_0). \tag{18}$$

Or, d'après $x'_n \in L$ pour $n = 1, 2, 3, \cdots$, on a $\vartheta(x'_n) \in E$ pour $n = 1, 2, 3, \cdots$, donc, d'après le définition de l'ensemble E:

$$\psi(\vartheta(x'_n)) = g(\varphi(\vartheta(x'_n))) = g(x'_n) \text{ pour } n = 1, 2, 3, \cdots,$$

ce qui donne, d'après (17) et (18):

$$\psi(u_0) = g(x_0). \tag{19}$$

Or, d'après $x_0 \in L$ on a $\vartheta(x_0) \in E$ donc, d'après la définition de l'ensemble E:

$$\psi(\vartheta(x_0)) = g(\varphi(\vartheta(x_0))) = g(x_0)$$

donc, d'après (19):

$$\psi(u_0) = \psi(\vartheta(x_0)). \tag{20}$$

Or, d'après (15) et $x_0 \in L \subset N$ on conclut que $\varphi(u_0) \in N$ et d'après (19) on trouve $\psi(u_0) \in g(L) = G \subset N$, donc $\psi(u_0) \in N$. Les nombres $\varphi(u_0)$ et $\psi(u_0)$ sont donc

irrationnels et on a les formules (16) et (20): il résulte donc de la propriété de la courbe (1)

$$u_0 = \vartheta(x_0)$$

contrairement à (11).

L'hypothèse que la formule (8) n'est pas vraie implique donc une contradiction. La formule (8) est donc vraie. Nous avons ainsi démontré que la fonction $\vartheta(x)$ est continue dans L. Il est ainsi établi que les ensembles E et L sont homéomorphes. Or, comme j'ai démontré,[10] un ensemble homéomorphe à un ensemble jouissant de la propriété de Baire[11] jouit de cette propriété. D'autre part, comme on sait, l'ensemble L ne jouit pas de la propriété de Baire: son homéomorphe E ne jouit non plus de la propriété de Baire et il existe un ensemble parfait P et un intervalle Δ dans lequel E est partout de deuxième catégorie par rapport à P, et par suite E est aussi une somme de deux ensembles disjoints $E = E_1 + E_2$ telle que les ensembles E_1 et E_2 sont chacun partout de deuxième catégorie sur Δ par rapport à P.

Définissons maintenant la fonction d'une variable réelle $\gamma(x)$ comme il suit:

$$\gamma(x) = 1 \text{ pour } x \in g(\varphi(E_1)) \tag{21}$$

et

$$\gamma(x) = 0 \text{ pour } x \text{ non } \in g(\varphi(E_1)). \tag{22}$$

On a évidemment (d'après $E_1 \subset E$):

$$E_x[\gamma(x) \neq 0] = g(\varphi(E_1)) \subset g(\varphi(E)) = g(L) = G$$

et, l'ensemble G étant toujours de première catégorie, on voit sans peine que la fonction $\varphi(x)$ jouit de la propriété de Baire.

Posons, pour x réels

$$f(x) = \gamma(\psi(x)), \tag{23}$$

c'est donc une fonction jouissant de la propriété de Baire d'une fonction continue (d'une variable réelle).

Si $x \in E_1$ on a, d'après $E_1 \subset E$, $x \in E$, donc, d'après la définition de l'ensemble E:

$$\varphi(x) \in L \text{ et } \psi(x) = g(\varphi(x)) \tag{24}$$

donc

$$\gamma(\psi(x)) = \gamma(g(\varphi(x))) \tag{25}$$

et, d'après $x \in E_1$ et (21), $\gamma(g(\varphi(x))) = 1$, donc, d'après (25) et (23): $f(x) = 1$.

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[10] Fundamenta Mathematicae t. IV, p. 319.

[11] On dit qu'un ensemble linéaire M jouit de la propriété de Baire, si pour tout ensemble parfait P il existe un intervalle d contenant à son intérieur des points de P et tel qu'au moins un des ensembles dPM et $dP - M$ est de 1re catégorie de Baire par rapport à P.

Si $x \in E_2$, on a, d'après $E_2 \subset E$, encore les formules (24) et (25), d'où, d'après $x \in E_2 \subset E$ et x non $\in E_1$ (puisque $E_1 E_2 = 0$), la fonction φ étant à valeurs distinctes dans E et la fonction g étant à valeurs distinctes dans $L = \varphi(E)$, donc la fonction $g(\varphi(x))$ étant à valeurs distinctes dans E, on trouve que $g(\varphi(x))$ est non $\in g(\varphi(E_1))$, et, d'après (22): $\gamma(g(\varphi(x))) = 0$, d'où, d'après (23) et (25): $f(x) = 0$.

On a donc

$$f(x) = 1 \text{ pour } x \in E_1 \text{ et } f(x) = 0 \text{ pour } x \in E_2$$

les ensembles E_1 et E_2 étant partout de deuxième catégorie par rapport à P dans l'intervalle Δ, il en résulte que la fonction $f(x)$ ne jouit pas de la propriété de Baire.

Notre théorème est ainsi démontré.[12]

Or, il est à remarquer qu'on démontre sans peine qu'une fonction continue d'une fonction jouissant de la propriété de Baire jouit toujours de cette propriété.

Or, d'après une remarque de M. Kuratowski[13] il résulte de l'hypothèse du continu qu'il existe une fonction continue de deux variables réelles $f(x, y)$ et une fonction d'une variable réelle $g(x)$ jouissant de la propriété de Baire, telles que la fonction $f(g(x), y)$ ne jouit pas de la propriété de Baire.

M. S. Ruziewicz a démontré,[14] que toute fonction d'une variable réelle est une fonction mesurable d'une fonction mesurable (même d'une fonction fixe de 1re classe de Baire). Or, un théorème analogue n'a pas lieu pour les fonctions jouissant de la propriété de Baire: il existe des fonctions d'une variable réelle qui ne sont pas des superpositions (d'un nombre fini) de fonctions jouissant de la propriété de Baire. P. e. on démontre sans peine que telles sont les fonctions discontinues sur tout ensemble parfait.

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[12] Une autre démonstration de ce théorème paraîtra dans mon livre *Hypothèse du continu* (Monografje Matematycsue t. IV) Warszahwa 1934, p. 75.

[13] Voir Fund. Math. t. XXII p. 56.

[14] Communication au IIe Congrès de Mathématiciens Roumains à Turnu Severin (Mai, 1932). Cf. aussi Mathematica t. VII (1933), p. 89.

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SEPARABLE SYSTEMS OF STÄCKEL

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In 1891 Stäckel[1] showed how to determine the quantities H_i in the Hamilton-Jacobi equation

$$\text{(A)} \qquad \sum_i \frac{1}{H_i^2}\left(\frac{\partial \varphi}{\partial x_i}\right)^2 + k^2(E - V)\varphi = 0,$$

so that the variables are separable, the solution being of the form ΣX_i, where X_i is a function of x_i alone. In 1893[2] he showed that when the quadratic differential form $\Sigma H_i^2 dx_i^2$ so determined is taken as the Riemannian metric of a space V_n the equations of the geodesics of V_n admit $n - 1$ independent quadratic first integrals other than the fundamental form. In §§1, 2 we show that when this condition is satisfied, the fundamental quadratic form is of the Stäckel type.

In 1927 Robertson[3] showed that for an equation of the form

$$\text{(B)} \qquad \sum_i H \frac{\partial}{\partial x_i}\left(\frac{H}{H_i^2}\frac{\partial \varphi}{\partial x_i}\right) + k^2(E - V)\varphi = 0, \qquad H = H_1 \cdots H_n$$

to admit by separation of the variables a solution of the form ΠX_i, where X_i is a function of x_i alone, the functions H_i^2 must be of the Stäckel form and $V = \Sigma \frac{f(x_i)}{H_i^2}$, where $f(x_i)$ is an arbitrary function of x_i alone, just as in the case of equation (A) as shown by Stäckel. He found that in this case there is the additional condition

$$\text{(C)} \qquad \varphi = \Pi \frac{H_i}{\psi_i(x_i)},$$

where φ is the determinant of the Stäckel functions φ_{ij} and ψ_i is a function of x_i at most. In §2 we show that this condition is equivalent to the equations

$$R_{ij} = 0 \qquad (i \neq j)$$

in the given coordinate system, R_{ij} being the components of the Ricci tensor used by Einstein.

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[1] Habilitationsschrift, Halle.

[2] Comptes Rendus, vol. 116, pp. 485–487; cf. also, Ricci et Levi-Civita, Méthodes de calcul différentiel absolu, Math. Annalen, vol. 54 (1901), pp. 183, 184.

[3] Bemerkung über separierbare Systeme in der Wellenmechanik, Math. Annalen, vol. 98, pp. 749–752.

In §§3–7 we determine the various canonical forms for euclidean 3-space and find that in each case the coordinate surfaces are confocal quadrics including the cases when one or more families consists of planes, and that every type of confocal quadrics affords a solution; only the case of real surfaces has been considered. We have thus the only orthogonal systems of coordinates in which the three dimensional Schrödinger equation can be solved by separation of the variables.

In §8 we show that similar results hold for euclidean spaces of higher order and in §9 determine the Stäckel forms for a V_3 of constant Riemannian curvature because of their bearing on the problem for euclidean 4-space.

1. **Quadratic first integrals.** A necessary and sufficient condition that

$$a_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds} = \text{const.}$$

be a quadratic first integral of the equation of geodesics of a Riemannian V_n is that

$$a_{ij,\,k} + a_{jk,\,i} + a_{ki,\,j} = 0, \tag{1.1}$$

where a comma followed by an index indicates covariant differentiation; there is no loss in assuming that a_{ij} is symmetric in the indices.[4]

If ρ_i are the roots of the determinant equation

$$|\, a_{ij} - \rho g_{ij}\,| = 0, \tag{1.2}$$

the equations

$$(a_{ij} - \rho_h g_{ij})\lambda^i_{h|} = 0, \tag{1.3}$$

determine an orthogonal ennuple of contravariant vectors of components $\lambda^i_{h|}$, where h indicates the vector and i the component.[5] Ordinarily the vector-fields so defined are not normal in the sense that a vector field admits a family of hypersurfaces orthogonal to the vectors.

We assume that a_{ij} is such that these vector-fields are normal and that the hypersurfaces are taken as parametric; and we write the fundamental form thus

$$ds^2 = e_1H_1^2\,(dx_1)^2 + \cdots + e_nH_n^2\,(dx_n)^2, \tag{1.4}$$

where the e's are plus or minus one as the case may be. In this case $\lambda^i_{h|} = 0$ for $i \neq h$ and $a_{ij} = 0$ for $i \neq j$. Then equations (1.1) for $j = k = i$ and $j \neq i$, $k = j$ respectively reduce to

$$\begin{aligned} &\frac{\partial \log \sqrt{a_{ii}}}{\partial x_i} = \frac{\partial \log H_i}{\partial x_i}, \\ &\frac{\partial a_{ii}}{\partial x_j} - 4a_{ii}\frac{\partial \log H_i}{\partial x_j} + a_{jj}\frac{1}{H_j^2}\frac{\partial H_i^2}{\partial x_j} = 0, \end{aligned} \tag{1.5}$$

[4] Eisenhart, Riemannian Geometry, p. 129. Hereafter such a reference is of the form R. G., p. 129.

[5] R. G., p. 108.

and equations (1.1) for i, j, k different are satisfied identically. From the first set of (1.5) we have

$$a_{ii} = \rho_i H_i^2 , \tag{1.6}$$

where ρ_i, thus defined, is independent of x_i. The second set of (1.5) reduce to

$$\frac{\partial}{\partial x_j} \log \frac{\rho_i - \rho_j}{H_i^2} = 0, \tag{1.7}$$

from which it follows that $(\rho_i - \rho_j)/H_i^2$ is independent of x_j. Writing these results in the form

$$\frac{\partial \rho_i}{\partial x_j} = (\rho_i - \rho_j) \frac{\partial \log H_i^2}{\partial x_j}, \qquad \frac{\partial \rho_i}{\partial x_i} = 0, \tag{1.8}$$

and expressing the condition of integrability of this system of equations, we obtain

$$(\rho_i - \rho_j) \left(\frac{\partial^2 \log H_i^2}{\partial x_i \partial x_j} + \frac{\partial \log H_i^2}{\partial x_j} \frac{\partial \log H_j^2}{\partial x_i} \right) = 0$$

and

$$(\rho_j - \rho_k) \left(\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} - \frac{\partial \log H_i^2}{\partial x_j} \frac{\partial \log H_i^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_j} \frac{\partial \log H_j^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_k} \frac{\partial \log H_k^2}{\partial x_j} \right) = 0.$$

In order that (1.8) may admit a solution with all the ρ's different we must have

$$\frac{\partial^2 \log H_i^2}{\partial x_i \partial x_j} + \frac{\partial \log H_i^2}{\partial x_j} \frac{\partial \log H_j^2}{\partial x_i} = 0, \tag{1.9}$$

$$\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} - \frac{\partial \log H_i^2}{\partial x_j} \frac{\partial \log H_i^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_j} \frac{\partial \log H_j^2}{\partial x_k} + \frac{\partial \log H_i^2}{\partial x_k} \frac{\partial \log H_k^2}{\partial x_j} = 0. \tag{1.10}$$

Since these equations are consistent, it follows that, when they are satisfied, equations (1.8) are completely integrable. One solution is $\rho_i = \rho_j = a$, a constant. We denote by ρ_i^α (for $\alpha = 2, \cdots, n$) $n - 1$ other solutions such that the determinant of the n solutions is not zero. This may be indicated in the determinant form

$$| \rho_i^\alpha - \rho_j^\alpha | \neq 0, \tag{1.11}$$

where i is fixed, and $\alpha = 2, \cdots, n$; $j = 1, \cdots, n$; $j \neq i$. In this case the equations of the geodesics admit $n - 1$ quadratic first integrals whose coefficients are

$$a_{ii}^\alpha = \rho_i^\alpha H_i^2, \qquad a_{ij}^\alpha = 0. \tag{1.12}$$

2. **The Stäckel Form.** It is our purpose to show that the conditions of the preceding section determine the Stäckel form of the fundamental form of the V_n. To this end we denote by φ_{ij} n^2 functions such that their determinant φ is not zero, and we denote by φ^{ij} the cofactor of φ_{ij} in φ. We put

$$H_i^2 = \frac{\varphi}{\varphi^{i1}}, \qquad \rho_i^\alpha = \frac{\varphi^{i\alpha}}{\varphi^{i1}}, \tag{2.1}$$

and understand that the φ's are such that ρ_i^α are independent of x_i.

Also we put

$$b_{ij}^\alpha = \frac{\rho_i^\alpha - \rho_j^\alpha}{H_i^2} = \frac{\varphi^{j1}\varphi^{i\alpha} - \varphi^{i1}\varphi^{j\alpha}}{\varphi\varphi^{j1}}, \tag{2.2}$$

and have from §1 that b_{ij}^α are independent of x_j. We have that

$$\varphi^{j1}\varphi^{i\alpha} - \varphi^{i1}\varphi^{j\alpha} = \varphi M_{j1i\alpha},$$

where $M_{j1i\alpha}$ is the algebraic complement of $\varphi_{j1}\varphi_{i\alpha} - \varphi_{i1}\varphi_{j\alpha}$ in the determinant φ.[6] Consequently we have

$$\varphi^{j1}b_{ij}^\alpha = M_{j1i\alpha} \qquad (i, j = 1, \cdots, n; \alpha = 2, \cdots, n). \tag{2.3}$$

From the definition of $M_{j1i\alpha}$ we have

$$\varphi^{j\alpha} = \sum_{i\,(\neq j)} \varphi_{i1} M_{j1i\alpha},$$

and consequently

$$\frac{\varphi^{j\alpha}}{\varphi^{j1}} = \sum_{i\,(\neq j)} \varphi_{i1} b_{ij}^\alpha.$$

Differentiating with respect to x_j, we have

$$0 = \sum_{i\,(\neq j)} \frac{\partial \varphi_{i1}}{\partial x_j} b_{ij}^\alpha \qquad (\alpha = 2, \cdots, n; j = 1, \cdots, n).$$

For a given j the determinant of b_{ij}^α is not zero, in consequence of (1.11) and (2.2). Hence a function φ_{i1} is a function of x_i at most.

From (2.2) we have

$$\varphi^{j1}b_{ij}^\alpha = -\varphi^{i1}b_{ji}^\alpha. \tag{2.4}$$

In consequence of this result and (2.3) we have

$$\frac{M_{j1i\beta}}{M_{j1i2}} = \frac{b_{ij}^\beta}{b_{ij}^2} = \frac{b_{ji}^\beta}{b_{ji}^2} \equiv \sigma_{ij\beta} = \sigma_{ji\beta} \qquad \begin{pmatrix} \beta = 3, \cdots, n \\ i \neq j \end{pmatrix}. \tag{2.5}$$

Since the second term is independent of x_j and the third of x_i, it follows that $\sigma_{ij\beta}$ is independent of x_i and x_j. From the identities

$$\sum_{\alpha}^{2,\cdots,n} \varphi_{k\alpha} M_{j1i\alpha} = 0 \qquad (i, j, k \neq),$$

[6] Cf. Kowalewski, Einführung in die Determinantentheorie, p. 80.

we have with the aid of (2.5)

$$\varphi_{k2} + \sum_{\beta}^{3,\cdots,n} \varphi_{k\beta}\sigma_{ij\beta} = 0. \tag{2.6}$$

Differentiating with respect to x_i, we have

$$\frac{\partial \varphi_{k2}}{\partial x_i} + \sum_{\beta}^{3,\cdots,n} \frac{\partial \varphi_{k\beta}}{\partial x_i}\sigma_{ij\beta} = 0. \tag{2.7}$$

For a given i and k, there are $n-2$ equations (2.6) satisfied by the $n-1$ quantities $\varphi_{k2}, \cdots, \varphi_{kn}$; and these same equations are satisfied by the derivatives of these quantities with respect to x_i, hence we have

$$\frac{\partial \varphi_{k\alpha}}{\partial x_i} = \mu_{ik}\varphi_{k\alpha},$$

or

$$\frac{\partial}{\partial x_i}\left(\frac{\varphi_{k\alpha}}{\varphi_{k\gamma}}\right) = 0,$$

for $\gamma \neq \alpha$. Such equations hold for $i = 1, \cdots, n;\ i \neq k$. Hence we have

$$\varphi_{i\alpha} = e^{\nu_i}\psi_{i\alpha}, \tag{2.8}$$

where $\psi_{i\alpha}$ are functions of x_i at most and ν_i are to be determined.

From (2.3) we have

$$b^{\alpha}_{j1}M_{11i\alpha} = b^{\alpha}_{i1}M_{11j\alpha} \qquad (i, j = 2, \cdots, n).$$

Substituting from (2.8) we obtain

$$e^{\nu_j}b^{\alpha}_{j1}N_{1i} = e^{\nu_i}b^{\alpha}_{i1}N_{1j},$$

where N_{1i} is independent of x_1 and x_i. Differentiating with respect to x_1, we have

$$\frac{\partial}{\partial x_1}(\nu_i - \nu_j) = 0 \qquad (i, j = 2, \cdots, n).$$

Again from (2.3) and (2.4) we have

$$b^{\alpha}_{j\alpha}M_{\alpha 1\beta\beta} + b^{\beta}_{\beta\alpha}M_{j1\alpha\alpha} = 0 \qquad \begin{pmatrix}\alpha, \beta = 2, \cdots, n;\ \alpha \neq \beta \\ j = 1, \cdots, n;\ j \neq \alpha\end{pmatrix}.$$

Substituting from (2.8) we have

$$e^{\nu_j}b^{\alpha}_{j\alpha}N_{\alpha\beta} + e^{\nu_\beta}b^{\beta}_{\beta\alpha}N_{\alpha j} = 0.$$

Differentiating with respect to x_α, we have

$$\frac{\partial}{\partial x_\alpha}(\nu_j - \nu_\beta) = 0.$$

Combining these results, we have

$$\frac{\partial(\nu_i - \nu_j)}{\partial x_k} = 0 \qquad (i, j, k = 1, \cdots, n; i, j, k \neq).$$

From the preceding equations we have

$$\nu_i - \nu_j = f_{ij}, \qquad \nu_i - \nu_k = f_{ik}, \qquad \nu_j - \nu_k = f_{jk},$$

where f_{ij} is at most a function of x_i and x_j. From these equations we have

$$f_{ij} - f_{ik} + f_{jk} = 0.$$

Differentiating with respect to x_i, we obtain

$$\frac{\partial f_{ij}}{\partial x_i} = \frac{\partial f_{ik}}{\partial x_i}.$$

Since the first term does not involve x_k and the second x_j, we have

$$f_{ij} = \sigma_i - \sigma_j, \qquad f_{ik} = \sigma_i - \sigma_k,$$

where σ_i is a function of x_i alone. Hence the above set of equations may be replaced by

$$\nu_i = \nu + \sigma_i,$$

where ν is undetermined, and (2.8) becomes

$$\varphi_{i\alpha} = e^{\nu}\theta_{i\alpha} \qquad (i = 1, \cdots, n; \alpha = 2, \cdots, n), \tag{2.9}$$

where $\theta_{i\alpha}$ are functions of x_i alone. Since we have shown that φ_{i1} is a function of x_i alone, it follows that when the above expressions are substituted in (2.1) the factor e^{ν} disappears, and consequently the general solution of the problem is obtained when each function φ_{ij} is a function of x_i alone, which is the Stäckel form for the separation of the variables. Hence we have:

A necessary and sufficient condition that the fundamental quadratic form of V_n can be given the Stäckel form is that the equations of geodesics admit $n - 1$ independent quadratic first integrals, that the roots of the characteristic equations (1.2) *for each of these integrals be simple, that* (1.11) *hold, and that the vector-fields determined by* (1.3) *be normal and be the same vector-fields for all the first integrals.* Also we have:

A necessary and sufficient condition that (1.4) *is in the Stäckel form is that equations* (1.9) *and* (1.10) *be satisfied.*[7]

We have yet to consider the condition (C) of the Introduction for the case of equation (B). To this end we observe that from the definition of φ and (2.1) we have

$$\frac{\partial \log \varphi}{\partial x_j} = \frac{1}{H_j^2}(\varphi'_{j1} + \rho_j^2 \varphi'_{j2} + \cdots + \rho_j^n \varphi'_{jn}),$$

[7] Cf. Dall'Acqua, Le equazioni di Hamilton-Jacobi che si integrano per separazione di variabili, Rend. di Palermo, vol. 33 (1912), pp. 341–351.

where the prime indicates differentiation. In consequence of (1.8) we have

$$\frac{\partial^2 \log \varphi}{\partial x_j \partial x_k} = -\frac{1}{H_j^2}\frac{\partial \log H_j^2}{\partial x_k}\left(\varphi'_{j1} + \sum_{\alpha}^{2\cdots n} \rho_k^\alpha \varphi'_{j\alpha}\right). \tag{2.10}$$

If we differentiate with respect to x_j the identity

$$\varphi_{j1} + \sum_{\alpha} \rho_k^\alpha \varphi_{j\alpha} = 0,$$

we have in consequence of (1.8)

$$\varphi'_{j1} + \sum_{\alpha} \rho_k^\alpha \varphi'_{j\alpha} = \sum_{\alpha} \varphi_{j\alpha}(\rho_j^\alpha - \rho_k^\alpha)\frac{\partial \log H_k^2}{\partial x_j} = \left(\varphi_{j1} + \sum_{\alpha} \varphi_{j\alpha}\rho_j^\alpha\right)\frac{\partial \log H_k^2}{\partial x_j}$$

$$= \frac{\varphi}{\varphi^{j1}}\frac{\partial \log H_k^2}{\partial x_j} = H_j^2 \frac{\partial \log H_k^2}{\partial x_j}.$$

Consequently we have from (2.10)

$$\frac{\partial^2 \log \varphi}{\partial x_j \partial x_k} = -\frac{\partial \log H_j^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j}.$$

From this and (1.9) we have

$$\frac{\partial^2 \log \varphi}{\partial x_j \partial x_k} = \frac{\partial^2 \log H_j^2}{\partial x_j \partial x_k} = \frac{\partial^2 \log H_k^2}{\partial x_j \partial x_k}. \tag{2.11}$$

In order that (C) be satisfied it is necessary and sufficient that

$$\frac{\partial^2 \log}{\partial x_j \partial x_k}\frac{\Pi H_i}{\varphi} = 0 \qquad (j, k = 1, \cdots, n; j \neq k),$$

which because of (2.11) may be written

$$\frac{\partial^2 \log}{\partial x_j \partial x_k}\Pi' H_i = 0, \tag{2.12}$$

where Π' indicates the product of the H's except H_j and H_k.

In order to give an interpretation to (2.12) we consider the expression for the Riemannian symbol R_{jiik} for i, j, k different, namely[8]

$$R_{jiik} = \frac{e_i H_i^2}{4}\left[2\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_i^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial \log H_j^2}{\partial x_k} - \frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_k^2}{\partial x_j}\right]. \tag{2.13}$$

In consequence of (1.10) this may be written

$$R_{jiik} = \frac{3}{4} e_i H_i^2 \frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k}. \tag{2.14}$$

[8] R. G., p. 119.

Since

$$g^{ii} = \frac{e_i}{H_i^2}, \qquad g^{ij} = 0 \qquad (i \neq j),$$

we have

$$R_{jk} = g^{il} R_{jilk} = \frac{3}{4} \frac{\partial^2 \log \Pi' H_i^2}{\partial x_j \partial x_k},$$

and consequently (2.12) is equivalent to

$$R_{jk} = 0 \qquad (j \neq k). \tag{2.15}$$

Consequently the condition (C) is satisfied by any Stäckel form of a euclidean space and of a space of constant Riemannian curvature. Incidentally we remark that the Schwarzschild form[9] for an Einstein 4-space satisfies the conditions (1.9) and (1.10) and in this case (C) is satisfied.

3. **Stäckel systems in euclidean space.** The components of the Riemannian tensor formed with respect to the quadratic form (1.4) are given by (2.13) and[10]

$$R_{ijkl} = 0 \tag{3.1}$$

$$R_{jiij} = e_i H_i^2 \left(\frac{\partial^2 \log H_i}{\partial x_j^2} + \frac{\partial \log H_i}{\partial x_j} \frac{\partial}{\partial x_j} \log \frac{H_i}{H_j} \right) + e_j H_j^2 \left(\frac{\partial^2 \log H_j}{\partial x_i^2} \right. \tag{3.2}$$

$$\left. + \frac{\partial \log H_j}{\partial x_i} \frac{\partial}{\partial x_i} \log \frac{H_j}{H_i} \right) + \sum_{k(\neq i,j)} e_i e_j e_k \frac{H_i^2 H_j^2}{H_k^2} \frac{\partial \log H_i}{\partial x_k} \frac{\partial \log H_j}{\partial x_k}.$$

From (1.9) we have

$$\frac{\partial^2}{\partial x_i \partial x_j} \log \frac{H_i^2}{H_j^2} = 0,$$

from which it follows that

$$H_i^2 = \varphi_{ij}^2 \theta_{ij}, \qquad H_j^2 = \varphi_{ji}^2 \theta_{ij}, \tag{3.3}$$

where φ_{ij} is independent of x_j and φ_{ji} of x_i. Substituting in (1.9), we find that

$$\theta_{ij} = \tau_{ij} + \tau_{ji}, \tag{3.4}$$

where τ_{ij} is independent of x_j and τ_{ji} of x_i.

Equating to zero the right-hand member of (2.14), we have in consequence of (1.10)

$$\frac{\partial^2 \log H_i^2}{\partial x_j \partial x_k} = 0, \tag{3.5}$$

$$\frac{\partial \log H_i}{\partial x_j} \frac{\partial \log H_i}{\partial x_k} - \frac{\partial \log H_i}{\partial x_j} \frac{\partial \log H_j}{\partial x_k} - \frac{\partial \log H_i}{\partial x_k} \frac{\partial \log H_k}{\partial x_j} = 0 \tag{3.6}$$

$$(i, j, k \neq).$$

[9] R. G., p. 93.
[10] R. G., p. 119.

Substituting in (3.5) from (3.3) and (3.4), we find that

$$\frac{\partial \tau_{ji}}{\partial x_j} = (\tau_{ij} + \tau_{ji})\psi_{ji}(x_i, x_j), \tag{3.7}$$

and similarly

$$\frac{\partial \tau_{ij}}{\partial x_i} = (\tau_{ij} + \tau_{ji})\psi_{ij}(x_i, x_j). \tag{3.8}$$

Differentiating these equations with respect to x_i and x_j respectively, we have

$$\frac{\partial \psi_{ji}}{\partial x_i} + \psi_{ji}\psi_{ij} = 0, \qquad \frac{\partial \psi_{ij}}{\partial x_j} + \psi_{ji}\psi_{ij} = 0.$$

Accordingly we have

$$\psi_{ji} = \frac{\partial \log \alpha}{\partial x_j}, \qquad \psi_{ij} = \frac{\partial \log \alpha}{\partial x_i},$$

and we find that $\alpha = \alpha_i + \alpha_j$, where α_i and α_j are functions of x_i and x_j respectively. Then from (3.7) and (3.8) it follows that

$$\tau_{ij} + \tau_{ji} = (\alpha_i + \alpha_j)\omega_{ij},$$

where ω_{ij} is independent of x_i and x_j. In consequence of this result and (3.3) it follows that

$$H_i^2 = X_i \prod_{j\,(\neq i)} (\sigma_{ij} + \sigma_{ji}) \qquad (i = 1, \cdots, n), \tag{3.9}$$

where σ_{ij} is a function of x_i at most and σ_{ji} of x_j at most. These expressions satisfy (1.9). In order that (3.6) be satisfied we must have

$$\sigma'_{ji}\sigma'_{ki}(\sigma_{jk} + \sigma_{kj}) - \sigma'_{ji}\sigma'_{kj}(\sigma_{ki} + \sigma_{ik}) - \sigma'_{ki}\sigma'_{jk}(\sigma_{ij} + \sigma_{ji}) = 0, \tag{3.10}$$

where a prime indicates the derivative, and permuting the indices cyclically we have

$$\begin{aligned} &\sigma'_{kj}\sigma'_{ij}(\sigma_{ki} + \sigma_{ik}) - \sigma'_{kj}\sigma'_{ik}(\sigma_{ij} + \sigma_{ji}) - \sigma'_{ij}\sigma'_{ki}(\sigma_{jk} + \sigma_{kj}) = 0, \\ &\sigma'_{ik}\sigma'_{jk}(\sigma_{ij} + \sigma_{ji}) - \sigma'_{ik}\sigma'_{ji}(\sigma_{jk} + \sigma_{kj}) - \sigma'_{jk}\sigma'_{ij}(\sigma_{ki} + \sigma_{ik}) = 0. \end{aligned} \tag{3.11}$$

Equating to zero the determinant of these equations, we have

$$\sigma'_{ij}\sigma'_{jk}\sigma'_{ki} + \sigma'_{ji}\sigma'_{kj}\sigma'_{ik} = 0. \tag{3.12}$$

The same result follows, if we differentiate the above equations with respect to x_i, x_j and x_k respectively.

On the assumption that none of the terms in (3.12) is zero it follows that $\sigma'_{ij}/\sigma'_{ik}$ is a constant. Accordingly we put

$$\sigma_{ij} = a_{ij}\sigma_i,$$

where a_{ij} is a constant and σ_i involves x_i at most. The constants must satisfy the relations

$$a_{ij}a_{jk}a_{ki} + a_{ji}a_{kj}a_{ik} = 0. \tag{3.13}$$

Equations (3.10) and (3.11) are satisfied in consequence of (3.13). Hence in this case we have

$$H_i^2 = X_i \prod_{j\,(\neq i)} (a_{ij}\sigma_i + a_{ji}\sigma_j).$$

If we put

$$\sigma_i = a_{jk}a_{ki}\bar{\sigma}_i, \qquad \sigma_j = a_{kj}a_{ik}\bar{\sigma}_j,$$

in consequence of (3.13) we have $a_{ij}\sigma_i + a_{ji}\sigma_j = a_{ij}a_{jk}a_{ki}(\bar{\sigma}_i - \bar{\sigma}_j)$ and the constant factor may be absorbed in X_i. Then we have in all generality $a_{ij} = -a_{ji} = 1$, and (3.13) becomes

$$a_{jk}a_{ki} - a_{kj}a_{ik} = 0.$$

If now we put $a_{ki}\sigma_k = -a_{ik}\bar{\sigma}_k$, we have

$$a_{ki}\sigma_k + a_{ik}\bar{\sigma}_i = a_{ik}(\bar{\sigma}_i - \bar{\sigma}_k),$$

so that in all generality we may take $a_{ki} = -1$, $a_{ik} = +1$. Then $a_{jk}\bar{\sigma}_j + a_{kj}\bar{\sigma}_k = a_{jk}(\bar{\sigma}_j - \bar{\sigma}_k)$, and thus $a_{jk} = -a_{kj} = 1$, and we have

$$H_i^2 = X_i \prod_{j\,(\neq i)} (\sigma_i - \sigma_j). \tag{3.14}$$

We consider now the cases which can arise when some of the σ's are constant. Suppose that $\sigma_{ij} = a_{ij}$, where a_{ij} is a constant. From the first of (3.11) it follows that either $\sigma_{ik} = a_{ik}$, or $\sigma_{kj} = a_{kj}$, the a's being constant; we use this notation for the present. If $\sigma_{ik} = a_{ik}$, the second of (3.11) is satisfied and there remains (3.10). This is satisfied in the following cases

$$\text{(i)}\ \sigma_{ji} = a_{ji},\ \sigma_{jk} = a_{jk}; \qquad \text{(ii)}\ \sigma_{ji} = a_{ji},\ \sigma_{ki} = a_{ki}; \qquad \text{(iii)}\ \sigma_{ki} = a_{ki},\ \sigma_{kj} = a_{kj}. \tag{3.15}$$

The last follows from (i) when j and k are interchanged. If σ_{ji} and σ_{ki} are not constants, we write (3.10) in the form

$$\sigma_{jk} + \sigma_{kj} - \frac{\sigma'_{kj}}{\sigma'_{ki}}(\sigma_{ki} + a_{ik}) - \frac{\sigma'_{jk}}{\sigma'_{ji}}(\sigma_{ji} + a_{ij}) = 0. \tag{3.16}$$

From this we have

$$\sigma_{jk} - \frac{\sigma'_{jk}}{\sigma'_{ji}}(\sigma_{ji} + a_{ij}) = c, \qquad \sigma_{kj} - \frac{\sigma'_{kj}}{\sigma'_{ki}}(\sigma_{ki} + a_{ik}) = -c,$$

where c is a constant, and consequently

$$\sigma_{ji} + a_{ij} = b(\sigma_{jk} - c), \qquad \sigma_{ki} + a_{ik} = d(\sigma_{kj} + c),$$

where b and d are constants. Hence $\sigma'_{ji} = b\sigma'_{jk}$, $\sigma'_{ki} = d\sigma'_{kj}$, so that we may put $\sigma_{ji} = a_{ji}\sigma_j$, $\sigma_{jk} = a_{jk}\sigma_j$; $\sigma_{ki} = a_{ki}\sigma_k$, $\sigma_{kj} = a_{kj}\sigma_k$, and then from (3.16) we have (3.13). Thus we have three distinct types:

(3.17) $\sigma_{ij} = a_{ij}$, $\sigma_{ji} = a_{ji}$, $\sigma_{ik} = a_{ik}$, $\sigma_{jk} = a_{jk}$;

(3.18) $\sigma_{ij} = a_{ij}$, $\sigma_{ji} = a_{ji}$, $\sigma_{ik} = a_{ik}$, $\sigma_{ki} = a_{ki}$;

(3.19) $\sigma_{ij} = a_{ij}$, $\sigma_{ik} = a_{ik}$, $\sigma_{ji} = a_{ji}\sigma_j$, $\sigma_{jk} = a_{jk}\sigma_j$, $\sigma_{ki} = a_{ki}\sigma_k$, $\sigma_{kj} = a_{kj}\sigma_k$.

In the first two cases the a's are arbitrary, in the last case they must satisfy (3.13).

When $\sigma_{kj} = a_{kj}$ and $\sigma_{ij} = a_{ij}$, we have from (3.10) and (3.11) the case (3.15 iii), or

$$\sigma'_{ji}(\sigma_{jk} + a_{kj}) - \sigma'_{jk}(\sigma_{ji} + a_{ij}) = 0.$$

If $\sigma_{ji} = a_{ji}$, $\sigma_{jk} = a_{jk}$, we have (3.18) on interchanging i and j. Otherwise we have the type

(3.20) $$\sigma_{ij} = a_{ij},\ \sigma_{kj} = a_{kj},\ \sigma_{ji} = a_{ji}\sigma_j,\ \sigma_{jk} = a_{jk}\sigma_j,$$
$$a_{ji}a_{kj} - a_{jk}a_{ij} = 0.$$

For $n = 3$ and $i = 3, j = 2, k = 1$, we have from (3.9) and (3.17) that the coordinates can be chosen so that we have

(3.21) $$H_1 = 1, \qquad H_2 = \varphi_1, \qquad H_3 = \psi_1,$$

where φ_1 and ψ_1 are functions of x_1 at most.

For the case (3.20) we may in all generality take $a_{ij} = a_{kj} = 0$ and then from (3.9) for $i = 2, j = 1, k = 3$ by a suitable choice of coordinates we have

(3.22) $$H_1^2 = 1, \qquad H_2^2 = X_2\sigma_1(\sigma_{23} + \sigma_{32}), \qquad H_3^2 = X_3\sigma_1(\sigma_{23} + \sigma_{32}),$$

where σ_1 is a function of x_1 at most. When σ_1 is a constant, we have the case (3.18) for $i = 1, j = 2, k = 3$.

For the case (3.19) we may take $a_{ij} = a_{ik} = 0$ and for $i = 2, j = 1, k = 3$ we have in all generality

(3.23) $$H_1^2 = H_3^2 = \sigma_1 + e\sigma_3, \qquad H_2^2 = \sigma_1\sigma_3,$$

where $e = +1$ or -1, it being understood that σ_1 and σ_3 are positive.

Finally we have from (3.14) the case

(3.24) $$H_i^2 = X_i(x_i - x_j)(x_i - x_k) \qquad (i, j, k = 1, 2, 3;\ i, j, k \neq).$$

For the further determination of the functions appearing in (3.21), (3.22), (3.23), and (3.24) we have from (3.2) for all the e's equal to 1 the conditions

(3.25) $$\frac{1}{H_j^2}\left(2\frac{\partial^2 \log H_i^2}{\partial x_j^2} + \frac{\partial \log H_i^2}{\partial x_j}\frac{\partial}{\partial x_j}\log\frac{H_i^2}{H_j^2}\right) + \frac{1}{H_i^2}\left(2\frac{\partial^2 \log H_j^2}{\partial x_i^2}\right.$$

$$+\frac{\partial \log H_j^2}{\partial x_i}\frac{\partial}{\partial x_i}\log\frac{H_j^2}{H_i^2}\Big) + \frac{1}{H_k^2}\frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_j^2}{\partial x_k} = 0 \qquad (i, j, k \neq).$$

These determinations will be made in the following sections.

When $n = 3$, equations (2.12) are (3.5). Consequently we have:

A necessary and sufficient condition that equation (B) be solvable by separation of the variables for $n = 3$ is that the H's be of one of the forms (3.21), (3.22), (3.23), (3.24).

4. **Types I.** From (3.21) and (3.25) for $i = 1, j = 2$ and $j = 3$, we find that

$$\varphi_1 = ax_1 + b, \qquad \psi_1 = cx_1 + d,$$

where a, b, c, d are constants. Substituting in (3.25) for $i = 2, j = 3$, we have $\frac{\partial\varphi_1}{\partial x_1}\frac{\partial\psi_1}{\partial x_1} = 0$. We take $a = 0, b = 1$. If $c = 0$, we have the cartesian case

$$(\mathrm{I}_1) \qquad H_i^2 = 1 \qquad (i = 1, 2, 3).$$

If $c \neq 0$, we have by a suitable choice of coordinates

$$(\mathrm{I}_2) \qquad H_1^2 = H_2^2 = 1, \qquad H_3^2 = x_1^2.$$

In this case the transformations of coordinates is

$$(4.1) \qquad x = x_1 \cos x_3, \qquad y = x_1 \sin x_3, \qquad z = x_2,$$

and the coordinate surfaces are the planes $x/y = \text{const.}$, $z = \text{const.}$ and the circular cylinders $x^2 + y^2 = x_1^2$.

The Stäckel matrices for these respective forms are

$$\begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & -1 & -\frac{1}{x_1^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

5. **Types II.** In discussing the types (3.22) we consider first the case where σ_{32} is a constant, which may be taken equal to zero in all generality and by suitable choice of coordinates we have

$$H_1^2 = 1, \qquad H_2^2 = \varphi^2(x_1), \qquad H_3^2 = \varphi^2(x_1)\psi^2(x_2).$$

For $i = 1, j = 2, 3$ in (3.25) we have $\varphi = ax_1 + b$. For $i = 2, j = 3$ we have

$$\frac{\partial^2\psi}{\partial x_2^2} + a^2\psi = 0.$$

If $a = 0$, we have Types I. If $a \neq 0$, we may take $a = 1$, $b = 0$ and obtain the case of polar coordinates

$$(\mathrm{II}_1) \qquad H_1^2 = 1, \qquad H_2^2 = x_1^2, \qquad H_3^2 = x_1^2 \sin^2 x_2.$$

If neither σ_{23} nor σ_{32} is constant, we may choose the coordinates so as to have

$$(5.1) \qquad H_1^2 = 1, H_2^2 = X_2\sigma_1(x_2 - x_3), \qquad H_3^2 = X_3\sigma_1(x_2 - x_3).$$

For $i = 1, j = 2, 3$ in (3.25) we have $\sigma_1 = (ax_1 + b)^2$. For $i = 2, j = 3$ we obtain

$$(5.2) \quad 2\left(\frac{1}{X_2} + \frac{1}{X_3}\right) + (x_2 - x_3)\left[\left(\frac{1}{X_3}\right)' - \left(\frac{1}{X_2}\right)'\right] - 4a^2(x_2 - x_3)^3 = 0.$$

Differentiating with respect to x_2, we obtain

$$(5.3) \qquad \left(\frac{1}{X_2}\right)' + \left(\frac{1}{X_3}\right)' + (x_3 - x_2)\left(\frac{1}{X_2}\right)'' - 12a^2(x_2 - x_3)^2 = 0.$$

Differentiating again with respect to x_2, we have

$$\left(\frac{1}{X_2}\right)''' = -24a^2,$$

and consequently

$$(5.4) \qquad \frac{1}{X_2} = -4a^2x_2^3 + cx_2^2 + dx_2 + e \equiv f(x_2).$$

Substituting this expression in (5.3) we have

$$\left(\frac{1}{X_3}\right)' = 12a^2x_3^2 - 2cx_3 - d,$$

and then from (5.2) we have $\frac{1}{X_3} = -f(x_3)$. There are two cases to be considered as $a = 0$ and $a \neq 0$. In the former case, as is seen on substituting the above expressions in (1.4) for $n = 3$, there is no loss in taking $b = 1$ and in the second case $a = 1, b = 0$. Hence we have the two forms

$$(5.5) \qquad H_1^2 = 1, \qquad H_2^2 = \frac{x_2 - x_3}{f(x_2)}, \qquad H_3^2 = \frac{x_3 - x_2}{f(x_3)},$$
$$f(x_i) = cx_i^2 + dx_i + e,$$

$$(5.6) \qquad H_1^2 = 1, \qquad H_2^2 = \frac{x_1^2(x_2 - x_3)}{f(x_2)}, \qquad H_3^2 = \frac{x_1^2(x_3 - x_2)}{f(x_3)},$$
$$f(x_i) = -4x_i^3 + cx_i^2 + dx_i + e.$$

If in (5.5) we assume that $f(x) = 0$ has two distinct roots, by a suitable choice of the coordinates, the form may be written $f(x) = 4(x^2 - ax)$, where $a > 0$, and $x_2 > a > x_3 > 0$; then the expressions for H_2^2 and H_3^2 are positive. If we put

$$(5.7) \qquad x_2 - a/2 = \frac{1}{2}a \cosh 2\xi, \qquad x_3 - a/2 = \frac{1}{2}a \cos 2\eta,$$

we have, on replacing a by a^2

$$(\mathrm{II}_2) \qquad ds^2 = dx_1^2 + \frac{1}{2}a^2(\cosh 2\xi - \cos 2\eta)(d\xi^2 + d\eta^2),$$

in which case the coordinate transformation is

$$(5.8) \qquad x = x_1, \qquad y = a\cosh\xi\cos\eta, \qquad z = a\sinh\xi\sin\eta.$$

Hence the coordinate surfaces are the planes $x = \text{const.}$ and the confocal cylinders

$$(5.9) \qquad \frac{y^2}{\cos^2\eta} - \frac{z^2}{\sin^2\eta} = a^2, \qquad \frac{y^2}{\cosh^2\xi} + \frac{z^2}{\sinh^2\xi} = a^2.$$

No real case exists for which the two roots of $f(x) = 0$ in (5.5) are equal, nor when $f(x)$ is a constant. In case $c = 0$, we may take $f(x) = 4x$, and $x_2 > 0 > x_3$. If we put $x_2 = \xi^2$, $x_3 = -\eta^2$, we have

$$(\mathrm{II}_3) \qquad ds^2 = dx_1^2 + (\xi^2 + \eta^2)(d\xi^2 + d\eta^2),$$

in which case the coordinate transformation is

$$x = x_1,\ y = \tfrac{1}{2}(\xi^2 - \eta^2),\ z = \xi\eta,$$

so that the coordinate surfaces are the planes $x = \text{const.}$, and the confocal parabolic cylinders

$$(5.10) \qquad z^2 = \xi^2(\xi^2 - 2y), \qquad z^2 = \eta^2(2y + \eta^2).$$

If we write $f(x)$ in (5.6) in the form $4(a - x)(b - x)(c - x)$ with $a > b > c$ and put

$$\frac{a-b}{a-c} = k^2, \qquad \frac{b-c}{a-c} = k'^2, \qquad k^2 + k'^2 = 1,$$

and

$$x_2 = a + (b - a)\,\mathrm{sn}^2(\xi, k), \qquad x_3 = c + (b - c)\,\mathrm{sn}^2(\eta, k'),$$

where $\mathrm{sn}\,\theta$ is the elliptic function, the form (5.6) becomes

$$(\mathrm{II}_4) \qquad ds^2 = dx_1^2 + x_1^2[k^2\,\mathrm{cn}^2(\xi, k) + k'^2\,\mathrm{cn}^2(\eta, k')](d\xi^2 + d\eta^2).$$

The coordinate transformation is

$$(5.11) \qquad x = x_1\,\mathrm{dn}(\xi, k)\,\mathrm{sn}(\eta, k'), \qquad y = x_1\,\mathrm{sn}(\xi, k)\,\mathrm{dn}(\eta, k'),$$
$$z = x_1\,\mathrm{cn}(\xi, k)\,\mathrm{cn}(\eta, k'),$$

so that the coordinate surfaces are the spheres and cones with the equations

$$(5.12) \qquad x^2 + y^2 + z^2 = x_1^2, \qquad \frac{k^2x^2}{\mathrm{dn}^2(\xi, k)} - \frac{y^2}{\mathrm{sn}^2(\xi, k)} + \frac{z^2}{\mathrm{cn}^2(\xi, k)} = 0,$$
$$\frac{x^2}{\mathrm{sn}^2(\eta, k')} - \frac{k'^2y^2}{\mathrm{dn}^2(\eta, k')} - \frac{z^2}{\mathrm{cn}^2(\eta, k')} = 0.$$

It is readily shown that if two of the roots of $f(x) = 0$ are equal, or all are equal, there are no real solutions of the problem.

The Stäckel matrices for the forms (II_1), (II_2), (II_3) and (II_4) are respectively

$$\begin{vmatrix} 1 & -\frac{1}{x_1^2} & 0 \\ 0 & 1 & -\csc^2 x_2 \\ 0 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & -1 & 0 \\ 0 & a^2\cosh 2\xi & -1 \\ 0 & -a^2\cos 2\eta & 1 \end{vmatrix},$$

$$\begin{vmatrix} 0 & -1 & 0 \\ 0 & \xi^2 & -1 \\ 0 & \eta^2 & 1 \end{vmatrix}, \quad \begin{vmatrix} x_1^2 & -1 & 0 \\ 0 & k^2\,\mathrm{cn}^2(\xi, k) & -1 \\ 0 & k'^2\,\mathrm{cn}^2(\eta, k') & 1 \end{vmatrix}.$$

6. **Types III.** We consider next

$$H_1^2 = \sigma_1 + e\sigma_3, \qquad H_2^2 = \sigma_1\sigma_3, \qquad H_3^2 = \sigma_1 + e\sigma_3, \tag{6.1}$$

where e is $+1$ or -1, it being understood that σ_1 and σ_3 are positive.

Substituting in (3.25) for $i = 1, j = 2$, we have

$$2\left(\sigma_1'' - \frac{\sigma_1'^2}{\sigma_1}\right) + e\sigma_3\left(2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2}\right) + e\frac{\sigma_3'^2}{\sigma_3} = 0. \tag{6.2}$$

Differentiating with respect to x_3, we have

$$2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} + \left(\frac{\sigma_3'^2}{\sigma_3}\right)'\frac{1}{\sigma_3'} = 0.$$

Hence we have

$$2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} = c, \qquad \left(\frac{\sigma_3'^2}{\sigma_3}\right)' = -c\sigma_3',$$

where c is a constant. From the second we have

$$\sigma_3'^2 = -c\sigma_3^2 + d\sigma_3, \tag{6.3}$$

where d is a constant. Then from (6.2) we have

$$\sigma_1'^2 = c\sigma_1^2 + de\sigma_1. \tag{6.4}$$

These expressions satisfy (3.25) for $i = 1, j = 3$ and $i = 2, j = 3$, without imposing any conditions on c and d.

We consider first the case when $c = 0$, which is possible only when $e = 1$ and d positive, as we understand that the coordinates are real. Then we have

$$\sqrt{\sigma_i} = \tfrac{1}{2}\sqrt{d}\,x_i + f_i.$$

By a suitable choice of coordinates we have

$$H_1^2 = H_3^2 = x_1^2 + x_3^2, \qquad H_2^2 = x_1^2 x_3^2. \tag{III$_1$}$$

The transformation is

$$x = x_1 x_3 \cos x_2, \qquad y = x_1 x_3 \sin x_2, \qquad z = \tfrac{1}{2}(x_1^2 - x_3^2), \tag{6.5}$$

so that the coordinate surfaces are the planes $x/y = \text{const.}$ and

$$x^2 + y^2 = x_1^2(x_1^2 - 2z), \qquad x^2 + y^2 = x_3^2(x_3^2 + 2z). \tag{6.6}$$

When $c \neq 0$, there is no loss of generality in assuming it to be positive, and replacing it by $4c^2$. We consider first the case when $e = 1$, and replace σ_1 and σ_3 by $\frac{\sigma_1 d}{4c^2}$ and $\frac{\sigma_3 d}{4c^2}$, noting from (6.3) that d is necessarily positive. Then the solution of (6.4) and (6.3) respectively is

$$\sigma_1 = \sinh^2(cx_1 + b), \qquad \sigma_3 = \sin^2(cx_3 + f).$$

By suitable choice of the coordinates we have

$$(\mathrm{III}_2) \quad H_1^2 = H_3^2 = a^2(\sinh^2 x_1 + \sin^2 x_3), \qquad H_2^2 = a^2 \sinh^2 x_1 \sin^2 x_3.$$

The transformation from cartesian coordinates is

$$x = a \sinh x_1 \sin x_3 \cos x_2, \qquad y = a \sinh x_1 \sin x_3 \sin x_2,$$
$$z = a \cosh x_1 \cos x_3.$$

The coordinate surfacesare the planes $y = x \tan x_2$ and the quadrics of revolution

$$\frac{x^2 + y^2}{\sinh^2 x_1} + \frac{z^2}{\cosh^2 x_1} = a^2, \qquad \frac{z^2}{\cos^2 x_3} - \frac{x^2 + y^2}{\sin^2 x_3} = a^2.$$

When $e = -1$, we obtain in like manner

$$(\mathrm{III}_3) \quad H_1^2 = H_3^2 = a^2(\sinh^2 x_1 + \cos^2 x_3), \qquad H_2^2 = a^2 \cosh^2 x_1 \sin^2 x_3.$$

The coordinate transformation is

$$x = a \cosh x_1 \sin x_3 \cos x_2, \qquad y = a \cosh x_1 \sin x_1 \sin x_2,$$
$$z = a \sinh x_1 \cos x_3,$$

and the coordinate surfaces are the planes $y = x \tan x_2$ and

$$\frac{x^2 + y^2}{\cosh^2 x_1} + \frac{z^2}{\sinh^2 x_1} = a^2, \qquad \frac{x^2 + y^2}{\sin^2 x_3} - \frac{z^2}{\cos^2 x_3} = a^2.$$

The Stäckel matrices for the above forms are respectively

$$\begin{vmatrix} x_1^2 & \frac{1}{x_1^2} & 1 \\ 0 & -1 & 0 \\ x_3^2 & \frac{1}{x_3^2} & -1 \end{vmatrix}, \quad \begin{vmatrix} a^2 \sinh^2 x_1 & \operatorname{csch}^2 x_1 & 1 \\ 0 & -1 & 0 \\ a^2 \sin^2 x_3 & \csc^2 x_3 & -1 \end{vmatrix}, \quad \begin{vmatrix} a^2 \cosh^2 x_1 & \operatorname{sech}^2 x_1 & -1 \\ 0 & 1 & 0 \\ -a^2 \sin^2 x_3 & -\csc^2 x_3 & 1 \end{vmatrix}.$$

7. **Types IV.** We consider finally the type (3.24), that is

$$H_i^2 = X_i(x_i - x_j)(x_i - x_k) \qquad (i, j, k \neq), \tag{7.1}$$

and we assume that $x_1 > x_2 > x_3$. When these expressions are substituted in (3.25) for $i = 1, j = 2$, we obtain

$$\frac{1}{X_3} + \frac{1}{(x_2 - x_1)^2}\left\{(x_3 - x_2)^2\left[(x_1 - x_3)\left(\frac{1}{X_1}\right)' - \left(\frac{2(x_3 - x_1)}{x_2 - x_1} + 1\right)\frac{1}{X_1}\right]\right.$$
$$\left. + (x_3 - x_1)^2\left[(x_2 - x_3)\left(\frac{1}{X_2}\right)' - \left(\frac{2(x_3 - x_2)}{x_1 - x_2} + 1\right)\frac{1}{X_2}\right]\right\} = 0. \tag{7.2}$$

Differentiating this equation with respect to x_2, we obtain a polynomial of the third degree in x_3. Each of its coefficients must vanish. Equating to zero the coefficient of x_3^3, we have

$$(x_1 - x_2)^2\left(\frac{1}{X_2}\right)'' + 4(x_1 - x_2)\left(\frac{1}{X_2}\right)' + 6\frac{1}{X_2} + 2(x_1 - x_2)\left(\frac{1}{X_1}\right)' - 6\frac{1}{X_1} = 0. \tag{7.3}$$

Differentiating with respect to x_2 twice we have

$$\left(\frac{1}{X_2}\right)^{\text{IV}} = 0,$$

and consequently

$$\frac{1}{X_2} = a_0 x_2^3 + a_1 x_2^2 + a_2 x_2 + a_3 \equiv f(x_2). \tag{7.4}$$

Substituting in (7.3) we find $\frac{1}{X_1} = f(x_1)$, and from (7.2) we have $\frac{1}{X_3} = f(x_3)$. These expressions satisfy the three conditions (3.25).

When $a_0 \neq 0$ in (7.4) and the roots of $f(x) = 0$ are distinct we write

$$f(x) = 4(\alpha - x)(\beta - x)(\gamma - x) \tag{7.5}$$

where $\alpha > \beta > \gamma$, and we have

$$H_i^2 = \frac{(x_i - x_j)(x_i - x_k)}{f(x_i)} \qquad (i, j, k \neq). \tag{IV$_1$}$$

This is the case of elliptic coordinates for which the transformation is

$$x^2 = \frac{(\alpha - x_1)(\alpha - x_2)(\alpha - x_3)}{(\alpha - \beta)(\alpha - \gamma)}, \qquad y^2 = \frac{(\beta - x_1)(\beta - x_2)(\beta - x_3)}{(\beta - \alpha)(\beta - \gamma)},$$
$$z^2 = \frac{(\gamma - x_1)(\gamma - x_2)(\gamma - x_3)}{(\gamma - \alpha)(\gamma - \beta)},$$

where $\alpha > x_1 > \beta > x_2 > \gamma > x_3$, the surfaces $x_i = \text{const.}$, being

$$\frac{x^2}{\alpha - x_i} + \frac{y^2}{\beta - x_i} + \frac{z^2}{\gamma - x_i} = 1.$$

It is readily shown that there is no possibility of a double or triple root of $f(x) = 0$ giving a real set of orthogonal surfaces for (7.1).

We consider next the case where $a_0 = 0$ in (7.4), and write $f(x) = 4(a - x)(b - x)$. In this case we have

$$(\mathrm{IV}_2) \qquad H_i^2 = \frac{(x_i - x_j)(x_i - x_k)}{f(x_i)}, \qquad f(x_i) = 4(a - x_i)(b - x_i), \qquad (i, j, k \neq),$$

and assume $x_1 > b > x_2 > a > x_3$. The transformation of coordinates is

$$x = \frac{x_1 + x_2 + x_3 - a - b}{2}, \qquad y^2 = \frac{(a - x_1)(a - x_2)(a - x_3)}{b - a},$$

$$z^2 = \frac{(b - x_1)(b - x_2)(b - x_3)}{a - b}, \tag{7.6}$$

so that the coordinate surfaces are the confocal paraboloids

$$\frac{y^2}{a - x_i} + \frac{z^2}{b - x_i} = x_i - 2x. \tag{7.7}$$

There is no real solution if the roots a, b are equal. Also there are no real solutions, when $a_0 = a_1 = 0$, and when $a_2 = 0$ also.

The Stäckel functions φ_{ij} are in these cases

$$\varphi_{i1} = \frac{x_i^2}{f(x_i)}, \qquad \varphi_{i2} = \frac{1}{f(x_i)}, \qquad \varphi_{i3} = \frac{x_i}{f(x_i)}.$$

The orthogonal systems of coordinate surfaces which afford separation of variables constitute the set of all real systems of confocal quadrics including the cases where one or more families of the systems consists of planes. Hence we have:

A necessary and sufficient condition that a triply orthogonal system of surfaces in euclidean 3-space be a coordinate system in terms of which the fundamental quadratic form of the space is such that the variables are separated in the corresponding Hamilton-Jacobi equation and the Laplace equation is that they be any system of confocal quadrics, including the cases when one or more families of the system consists of planes.

Also as a result of the preceding investigation we have:
Equations (1.9) *and* (1.10) *constitute a necessary and sufficient condition that the coordinate surfaces of a triply orthogonal system in euclidean 3-space be confocal quadrics, including the cases when one or more families of the system consists of planes.*

8. **Euclidean spaces of higher order.** When $n > 3$, we may analyze the various types as in the former discussion. We consider first the case when by a choice of x_1 we have $H_1 = 1$. In this case as follows from (3.9) we have without loss of generality, $\sigma_{j1} = 0$ for $j > 1$. From (3.9) and (3.20) for $i = 2, j = 1$, $k > 2$ it follows that

$$H_j^2 = X_j \sigma_{1j} \prod_k (\sigma_{jk} + \sigma_{kj}) \qquad (j, k = 2, \cdots, n; j \neq k),$$

where σ_{1j} is a function of x_1 at most. From (3.25) for $i = 1, j > 1$ it follows that $\sigma_{1j} = (a_j x_1 + b_j)^2$, where a_j and b_j are constants. For the case when all the a's are equal and all the b's, by a suitable choice of x_1 we have either $\sigma_{1j} = x_1^2$ or $\sigma_{1j} = c^2$, where c is a constant. If we substitute the above expressions in (3.2) for $i, j > 1$, we find that the hypersurfaces are of constant curvature $1/x_1^2$ in the first case and euclidean in the second. Furthermore, since $H_1 = 1$, the spaces $x_1 = \text{const.}$ are geodesically parallel. Consequently in the first case we have concentric hyperspheres and in the second case parallel hyperplanes. When $x_1 = \text{const.}$ are concentric hyperspheres, any other system of hypersurfaces consists of a pencil of planes whose axis passes through the common center of the hyperspheres or of quadric hypercones with vertices at the common center. The character of these other hypersurfaces depends upon the possible forms of $H_2^2\,dx_2^2 + \cdots + H_n^2\,dx_n^2$; in §9 we classify the Stäckel types for a V_3 of constant positive curvature. When the hyperplanes $x_1 = \text{const.}$ are parallel, any other system of hypersurfaces consists of hyperplanes orthogonal to the former or of coaxial quadric cylinders; the situation for $n = 4$ is readily obtained by giving to $H_2^2\,dx_2^2 + \cdots + H_4^2\,dx_4^2$ the various forms in §§4–7. These types for which $H_1 = 1$ are generalizations of those discussed in §§4, 5.

Generalizations of type III arise when we take $\sigma_{1j} = 0$ $(j > 1)$ and none of σ_{j1} is constant. In this case as follows from (3.19)

$$H_1^2 = \sigma_2 \cdots \sigma_n, \qquad H_j^2 = \prod_k (a_{jk}\sigma_j + a_{kj}\sigma_k)$$

$$(j, k = 2, \cdots, n; j \neq k).$$

From the equation obtained by equating to zero the right-hand member of (3.2) for $i, j = 2, \cdots, n$ we obtain a result which shows by means of (5.2) that the hypersurfaces $x_1 = \text{const.}$ are hyperplanes. Since x_1 does not appear in the H's, the sections by all these hyperplanes of each hypersurface $x_j = \text{const.}$ are the same. Analogously to types III these hyperplanes form a pencil and the hypersurfaces $x_j = \text{const.}$ are generalized spaces of revolution.

When i, j in (3.14) take the values $1, \cdots, n$, $\sigma_i = x_i$, and

$$\frac{1}{X_i} = 4(a_1 - x_i) \cdots (a_n - x_i),$$

we have the general elliptic coordinates in euclidean n-space discussed by Jacobi.[11] If $\bar{x}_i$ are cartesian coordinates, the transformation is

$$\bar{x}_i^2 = \frac{\prod_j (a_i - x_j)}{\prod_{j(\neq i)} (a_i - a_j)}.$$

[11] Vorlesungen über Dynamik, Berlin 1866, pp. 198–205.

9. **Stäckel systems in 3-space of constant curvature $1/c^2$.** If the curvature of V_3 is $1/c^2$, we have in place of (3.25)

$$\frac{1}{H_j^2}\left(2\frac{\partial^2 \log H_i^2}{\partial x_j^2}+\frac{\partial \log H_i^2}{\partial x_j}\frac{\partial}{\partial x_j}\log\frac{H_i^2}{H_j^2}\right) \tag{9.1}$$

$$+\frac{1}{H_i^2}\left(2\frac{\partial^2 \log H_j^2}{\partial x_i^2}+\frac{\partial \log H_j^2}{\partial x_i}\frac{\partial}{\partial x_i}\log\frac{H_j^2}{H_i^2}\right)+\frac{1}{H_k^2}\frac{\partial \log H_i^2}{\partial x_k}\frac{\partial \log H_j^2}{\partial x_k}=-\frac{4}{c^2}.$$

If we write (3.21) in the form

$$H_1^2=c^2,\qquad H_2^2=c^2\varphi_1^2,\qquad H_3^2=c^2\psi_1^2,$$

we have from (9.1) for $i=1, j=2$ and $i=1, j=3$ that φ_1 and ψ_1 must satisfy

$$\frac{\partial^2\theta}{\partial x_1^2}+\theta=0,$$

and from (9.1) for $i=2, j=3$ we have

$$\frac{\partial\varphi_1}{\partial x_1}\frac{\partial\psi_1}{\partial x_1}=-\varphi_1\psi_1.$$

Accordingly by a suitable choice of x_1 we have

$$ds^2=c^2(dx_1^2+\sin^2 x_1\,dx_2^2+\cos^2 x_1 dx_3^2). \tag{I}$$

We consider next the case

$$H_1^2=c^2,\qquad H_2^2=c^2\varphi^2(x_1),\qquad H_3^2=c^2\varphi^2(x_1)\psi^2(x_2).$$

From (9.1) for $i=1, j=2$ and 3 we have

$$\frac{\partial^2\varphi}{\partial x_1^2}+\varphi=0,$$

and hence

$$\varphi=a\sin(x_1+b).$$

From (9.1) for $i=2, j=3$ we have

$$\frac{\partial^2\psi}{\partial x_2^2}=-a^2\psi$$

and consequently

$$\psi=d\sin(ax_2+e).$$

By a suitable choice of coordinates we have

$$ds^2=c^2\,[dx_1^2+\sin^2 x_1\,(dx_2^2+\sin^2 x_2\,dx_3^2)]. \tag{II}$$

For the case

$$H_1^2=c^2,\qquad H_2^2=c^2X_2\sigma_1^2(x_2-x_3),\qquad H_3^2=c^2X_3\sigma_1^2(x_2-x_3),$$

we find that for $i = 1, j = 2, 3$, we have

$$\sigma_1 = a \sin (x_1 + b).$$

By a change of x_1 and replacing X_2 by X_2/a^2 we have

$$H_1^2 = c^2, \qquad H_2^2 = c^2 X_2 \sin^2 x_1 (x_2 - x_3), \qquad H_3^2 = c^2 X_3 \sin^2 x_1 (x_2 - x_3).$$

Then for $i = 2, j = 3$, we get (5.2) with $a = 1$. Hence we have

(III) $$ds^2 = c^2 \{dx_1^2 + \sin^2 x_1 [k^2 \operatorname{cn}^2 (x_2, k) + k'^2 \operatorname{cn}^2 (x_3, k')] (dx_2^2 + dx_3^2)\}.$$

For the case

$$H_1^2 = c^2(\sigma_1 + e\sigma_3), \qquad H_2^2 = c^2\sigma_1\sigma_3, \qquad H_3^2 = c^2(\sigma_1 + e\sigma_3),$$

where e is $+1$ or -1, we have from (9.1) for $i = 1, j = 2$

(9.2) $$2\left(\sigma_1'' - \frac{\sigma_1'^2}{\sigma_1}\right) + e\sigma_3\left(2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2}\right) + e\frac{\sigma_3'^2}{\sigma_3} = -4(\sigma_1 + e\sigma_3)^2.$$

Differentiating with respect to x_3, we obtain

$$2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} + \left(\frac{\sigma_3'^2}{\sigma_3}\right)'\frac{1}{\sigma_3'} = -8(\sigma_1 + e\sigma_3).$$

Hence we have

$$2\frac{\sigma_1''}{\sigma_1} - \frac{\sigma_1'^2}{\sigma_1^2} + 8\sigma_1 = 4d, \qquad \left(\frac{\sigma_3'^2}{\sigma_3}\right)'\frac{1}{\sigma_3'} + 8e\sigma_3 = -4d,$$

where d is a constant, from which and (9.2) we have

$$\sigma_1'^2 = 4\sigma_1(f + d\sigma_1 - \sigma_1^2), \qquad \sigma_3'^2 = 4\sigma_3(ef - d\sigma_3 - e\sigma_3^2).$$

These expressions satisfy (9.1) for $i = 1, j = 3$ and $i = 2, j = 3$.

For $e = 1$, we have, on putting

$$f = a^2 b^2, \qquad d = b^2 - a^2, \qquad k = \frac{b}{\sqrt{a^2 + b^2}}, \qquad k' = \frac{a}{\sqrt{a^2 + b^2}},$$

$$\sigma_1 = b^2 \operatorname{cn}^2 (\sqrt{a^2 + b^2}\, x_1, k), \qquad \sigma_3 = a^2 \operatorname{cn}^2 (\sqrt{a^2 + b^2}\, x_3, k').$$

A real solution does not exist when $e = -1$. Hence we have by a proper choice of x_1 and x_3

(IV) $$H_1^2 = H_3^2 = c^2 [k^2 \operatorname{cn}^2 (x_1, k) + k'^2 \operatorname{cn}^2 (x_3, k')],$$
$$H_2^2 = c^2 a^2 b^2 \operatorname{cn}^2 (x_1, k) \operatorname{cn}^2 (x_3, k').$$

For the case

$$H_1^2 = c^2 X_1 (x_1 - x_2)(x_1 - x_3), \qquad H_2^2 = c^2 X_2 (x_2 - x_1)(x_2 - x_3),$$
$$H_3^2 = c^2 X_3 (x_3 - x_1)(x_3 - x_2),$$

equation (9.1) for $i = 1, j = 2$ becomes

$$\frac{1}{X_3} + \frac{1}{(x_1 - x_2)^2}\left\{(x_3 - x_2)^2\left[(x_1 - x_3)\left(\frac{1}{X_1}\right)' - \left(\frac{2(x_3 - x_1)}{x_2 - x_1} + 1\right)\frac{1}{X_1}\right]\right.$$

$$\left. + (x_3 - x_1)^2\left[(x_2 - x_3)\left(\frac{1}{X_2}\right)' - \left(\frac{2(x_3 - x_2)}{x_1 - x_2} + 1\right)\frac{1}{X_2}\right]\right\} \tag{9.3}$$

$$+ 4(x_3 - x_1)^2 (x_3 - x_2)^2 = 0.$$

Differentiating with respect to x_2, we obtain a polynomial of the third degree in x_3. Each of its coefficients must vanish. Equating to zero the coefficient of x_3^3, we have

$$(x_1 - x_2)^2\left(\frac{1}{X_2}\right)'' + 4(x_1 - x_2)\left(\frac{1}{X_2}\right)' + 6\frac{1}{X_2} + 2(x_1 - x_2)\left(\frac{1}{X_1}\right)' - 6\frac{1}{X_1} \tag{9.4}$$

$$+ 8(x_1 - x_2)^4 = 0.$$

Differentiating twice with respect to x_2, we have

$$\left(\frac{1}{X_2}\right)^{IV} + 96 = 0,$$

are consequently

$$\frac{1}{X_2} = -4x_2^4 + a_1 x_2^3 + a_2 x_2^2 + a_3 x_2 + a_4 \equiv f(x_2).$$

Thus from (9.4) we have $1/X_1 = f(x_1)$ and from (9.3) $1/X_3 = f(x_3)$. Hence we have finally

$$H_i^2 = \frac{c^2 (x_i - x_j)(x_i - x_k)}{f(x_i)} \qquad (i, j, k \neq), \tag{V}$$

$$f(x_i) = -4x_i^4 + a_1 x_i^3 + a_2 x_i^2 + a_3 x_i + a_4.$$

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RIQUIER'S EXISTENCE THEOREMS[1]

BY JOSEPH MILLER THOMAS

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This paper is an addendum to an expository article previously published under the same title.[2] Its purpose is, in the first place, to remedy an omission from that paper (§1). Secondly, there is given for the complete set of monomials a definition which brings it in accord with Janet's definition,[3] at the same time maintaining the advantage of getting the complete and complementary sets directly from the Maclaurin expansion (§2). In §3 Riquier's method of reducing an orthonomic system to passive form is applied to deduce the general theorem for compatibility of a system of total differential equations. It thus becomes apparent that the method originally used in deducing this theorem is essentially Riquier's, and that the theorem itself is simply a statement of what his result for orthonomic systems becomes in a very special case. In §4 we rephrase (and at the same time generalize) a certain problem[4] so that its solution is seen at once to depend upon Riquier's theorems. Finally (§5) the opportunity is taken to point out one or two minor inaccuracies.

1. **The integrability conditions of an orthonomic system.** In §2 of the previous paper, the requirement was made that any monomial of a set E which is a multiple of another in the set be omitted before applying the process of forming the complete set. Some of the original monomials may therefore fail to be in the complete set. This is of no consequence in §4 because the theorem there stated only makes an assertion about the satisfaction of the equations whose left members are in the complete set.

In §5, however, if certain equations whose left members are derivatives of others are omitted, it is necessary to augment the integrability conditions by the conditions that the omitted equations be satisfied. It is these conditions that I failed to mention in §5 of my original treatment. Their formation is simple: in the omitted equations substitute for the principal derivatives their expressions in terms of the parametric.

Another way of covering the case in which one or more left members are derivatives of other left members is given in the following section. It is perhaps to be preferred to that just given. When it is adopted, the additional conditions mentioned above appear automatically among the integrability conditions.

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[1] Presented to the American Mathematical Society, November 27, 1931.

[2] These Annals, (2), vol. 30 (1929), pp. 285–310.

[3] M. Janet, Journal de mathématiques, (8), vol. 3 (1920), pp. 65–151.

[4] Thomas, T. Y., and Titt, E. W., these Annals, (2), vol. 34 (1933), pp. 40–65.

It is to be noted that the complementary set and, consequently, the form of the initial determinations are the same under the two methods of treatment.

2. **The complete set of monomials.** Let the process of separation described on pp. 286–87 of the former paper be carried out as indicated to give the expression

$$u = \Sigma x^{i_1} x^{i_2} \cdots x^{i_n} f_{i_1 i_2 \ldots i_n}. \tag{2.1}$$

Then consider a sub-expansion

$$x^{i_1} x^{i_2} \cdots x^{i_n} f_{i_1 i_2 \ldots i_n} \tag{2.2}$$

from (2.1) in conjunction with the set of monomials, other than $x^{i_1} x^{i_2} \cdots x^{i_n}$, which are in the original set E and which are terms in (2.2). Remove the common factor $x^{i_1} x^{i_2} \cdots x^{i_n}$ from these monomials, and separate $f_{i_1 i_2 \ldots i_n}$ with respect to the resulting monomials as was f with respect to E. If we repeat this on all sub-expansions (2.2) in (2.1) and on all sub-expansions arising from them in the process, after a finite number of steps we have a result of the form (2.1) such that every monomial in E is present among the $x^{i_1} x^{i_2} \cdots x^{i_n}$'s multiplying the f's. This is so because there is only a finite number of monomials E and it takes at most n steps to make one of them appear as an $x^{i_1} x^{i_2} \cdots x^{i_n}$. The complete set of monomials determined by E will be defined as the set of monomials $x^{i_1} x^{i_2} \cdots x^{i_n}$ in the final form (2.1) which are multiples of monomials in E. The complete set so defined contains all monomials of E.

As an example, consider that given on pp. 288–89 of the former paper. If E contains, in addition to the monomials there considered, the following

$$xy^2 z, \qquad xy^3 z^2, \tag{2.3}$$

the expression

$$xy^2 f_{120}(y, z)$$

must be further separated; that is, $f_{120}(y, z)$ must be separated with respect to the monomials

$$z, \qquad yz^2. \tag{2.4}$$

In the first step, yz^2 is omitted, being a multiple of the other, and we have

$$f_{120}(y, z) = f_{120}(y) + zf_{121}(y, z).$$

Since yz^2 is a monomial of (2.4) contained in a term of $zf_{121}(y, z)$, we must separate $f_{121}(y, z)$ with respect to yz. This gives

$$\begin{aligned} f_{121}(y, z) &= f_{121}(z) + y[f_{131}(y) + zf_{132}(y, z)] \\ &= f_{121}(z) + yf_{131}(y) + yzf_{132}(y, z). \end{aligned}$$

Hence the complete set, formed in accordance with the present definition and

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BY JOSEPH MILLER THOMAS

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This paper is an addendum to an expository article previously published under the same title.[2] Its purpose is, in the first place, to remedy an omission from that paper (§1). Secondly, there is given for the complete set of monomials a definition which brings it in accord with Janet's definition,[3] at the same time maintaining the advantage of getting the complete and complementary sets directly from the Maclaurin expansion (§2). In §3 Riquier's method of reducing an orthonomic system to passive form is applied to deduce the general theorem for compatibility of a system of total differential equations. It thus becomes apparent that the method originally used in deducing this theorem is essentially Riquier's, and that the theorem itself is simply a statement of what his result for orthonomic systems becomes in a very special case. In §4 we rephrase (and at the same time generalize) a certain problem[4] so that its solution is seen at once to depend upon Riquier's theorems. Finally (§5) the opportunity is taken to point out one or two minor inaccuracies.

1. **The integrability conditions of an orthonomic system.** In §2 of the previous paper, the requirement was made that any monomial of a set E which is a multiple of another in the set be omitted before applying the process of forming the complete set. Some of the original monomials may therefore fail to be in the complete set. This is of no consequence in §4 because the theorem there stated only makes an assertion about the satisfaction of the equations whose left members are in the complete set.

In §5, however, if certain equations whose left members are derivatives of others are omitted, it is necessary to augment the integrability conditions by the conditions that the omitted equations be satisfied. It is these conditions that I failed to mention in §5 of my original treatment. Their formation is simple: in the omitted equations substitute for the principal derivatives their expressions in terms of the parametric.

Another way of covering the case in which one or more left members are derivatives of other left members is given in the following section. It is perhaps to be preferred to that just given. When it is adopted, the additional conditions mentioned above appear automatically among the integrability conditions.

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[1] Presented to the American Mathematical Society, November 27, 1931.

[2] These Annals, (2), vol. 30 (1929), pp. 285–310.

[3] M. Janet, Journal de mathématiques, (8), vol. 3 (1920), pp. 65–151.

[4] Thomas, T. Y., and Titt, E. W., these Annals, (2), vol. 34 (1933), pp. 40–65.

It is to be noted that the complementary set and, consequently, the form of the initial determinations are the same under the two methods of treatment.

2. **The complete set of monomials.** Let the process of separation described on pp. 286–87 of the former paper be carried out as indicated to give the expression

$$u = \Sigma x^{i_1} x^{i_2} \cdots x^{i_n} f_{i_1 i_2 \ldots i_n}. \tag{2.1}$$

Then consider a sub-expansion

$$x^{i_1} x^{i_2} \cdots x^{i_n} f_{i_1 i_2 \ldots i_n} \tag{2.2}$$

from (2.1) in conjunction with the set of monomials, other than $x^{i_1} x^{i_2} \cdots x^{i_n}$, which are in the original set E and which are terms in (2.2). Remove the common factor $x^{i_1} x^{i_2} \cdots x^{i_n}$ from these monomials, and separate $f_{i_1 i_2 \ldots i_n}$ with respect to the resulting monomials as was f with respect to E. If we repeat this on all sub-expansions (2.2) in (2.1) and on all sub-expansions arising from them in the process, after a finite number of steps we have a result of the form (2.1) such that every monomial in E is present among the $x^{i_1} x^{i_2} \cdots x^{i_n}$'s multiplying the f's. This is so because there is only a finite number of monomials E and it takes at most n steps to make one of them appear as an $x^{i_1} x^{i_2} \cdots x^{i_n}$. The complete set of monomials determined by E will be defined as the set of monomials $x^{i_1} x^{i_2} \cdots x^{i_n}$ in the final form (2.1) which are multiples of monomials in E. The complete set so defined contains all monomials of E.

As an example, consider that given on pp. 288–89 of the former paper. If E contains, in addition to the monomials there considered, the following

$$xy^2 z, \qquad xy^3 z^2, \tag{2.3}$$

the expression

$$xy^2 f_{120}(y, z)$$

must be further separated; that is, $f_{120}(y, z)$ must be separated with respect to the monomials

$$z, \qquad yz^2. \tag{2.4}$$

In the first step, yz^2 is omitted, being a multiple of the other, and we have

$$f_{120}(y, z) = f_{120}(y) + z f_{121}(y, z).$$

Since yz^2 is a monomial of (2.4) contained in a term of $z f_{121}(y, z)$, we must separate $f_{121}(y, z)$ with respect to yz. This gives

$$\begin{aligned} f_{121}(y, z) &= f_{121}(z) + y[f_{131}(y) + z f_{132}(y, z)] \\ &= f_{121}(z) + y f_{131}(y) + yz f_{132}(y, z). \end{aligned}$$

Hence the complete set, formed in accordance with the present definition and

for the original set E augmented by (2.3), is that given in the former paper with the monomial xy^2 of multipliers y, z replaced by the following

$$\begin{aligned} &xy^2 && \text{with multiplier } y;\\ &xy^2z && \text{“ “ } z;\\ &xy^3z && \text{“ “ } y;\\ &xy^3z^2 && \text{“ multipliers } y, z. \end{aligned}$$

3. **Deduction of the theorem on total differential equations.** Consider the system

$$\frac{\partial u_\alpha}{\partial x_i} = \varphi_{\alpha i}(u, x) \qquad (\alpha = 1, 2, \cdots, r;\ i = 1, 2, \cdots, n), \tag{3.1}$$

in which the φ's are given, holomorphic functions. With the following cotes assigned

	u_α	x_i
1st	0	1
2nd	$r - \alpha$	0,

the system satisfies the definition given for orthonomic systems on p. 303 of the previous paper. The choice of second cotes is arbitrary and may be readjusted, if necessary, to make possible the notation employed below.

The separation (2.1) is made with respect to the set of monomials $x_1, \cdots, x_n$ for each unknown, and is

$$u = f_{00\cdots0} + x_1 f_{10\cdots0}(x_1, \cdots, x_n) + x_2 f_{01\cdots0}(x_2, \cdots, x_n) + \cdots + x_n f_{00\cdots1}(x_n).$$

The multipliers for any x are, accordingly, itself and the x's following it in the natural order of the indices. Hence the non-multipliers for any x are the x's which precede it in the natural order. The integrability conditions (§5 of the former paper) are therefore obtained by forming the equations

$$\frac{\partial \varphi_{\alpha i}}{\partial x_j} - \frac{\partial \varphi_{\alpha j}}{\partial x_i} = 0 \qquad (j < i = 1, 2, \cdots, n),$$

and eliminating the derivatives of the u's by means of (3.1). Consequently, they involve only the u's and the x's. We write them as

$$F_1. \tag{3.2}$$

Riquier's method of reducing (3.1) to passive form is as follows.[5] Relations (3.2) are solved for certain of the u's in terms of the others. By properly choos-

[5] See §7 of previous paper.

ing the second cotes of the unknowns, we may suppose them solved for $u_1, \cdots, u_k$:

$$u_\lambda = u_\lambda(u_{k+1}, \cdots, u_r; x) \qquad (\lambda = 1, 2, \cdots, k). \tag{3.3}$$

Equations (3.3) taken with (3.1) constitute a new orthonomic system. The complete set for $u_{k+1}, \cdots, u_r$ is as before. That for $u_1, \cdots, u_k$ is

$$1, x_1, \cdots, x_n,$$

and all the variables are non-multipliers for the monomial 1, which corresponds to the left member u. Hence the integrability conditions of the system (3.1) plus (3.3) are, in addition to those of (3.1) already expressed in (3.2), the results of differentiating the right members of (3.3) equated to the corresponding right members of (3.1) with the (principal) derivatives of $u_{k+1}, \cdots, u_r$ eliminated by means of the other equations in (3.1). This, of course, amounts to differentiating (3.3) and eliminating first derivatives by means of (3.1); or to performing the same operation on the unsolved form of (3.3), namely, (3.2).

We denote the system of integrability conditions thus obtained for (3.1) plus (3.3) by

$$F_2. \tag{3.4}$$

Equations (3.4) are now treated as were (3.2). Repetition gives a sequence of systems

$$F_1, F_2, F_3, \cdots. \tag{3.5}$$

If (3.1) is compatible, a passive system is reached and the equations in the corresponding F are identities. In other words, there exists a positive integer N such that the first N systems (3.5) are compatible in the u's and such that every solution of them satisfies the $(N+1)$th system F_{N+1}.

As we saw above, the system $F_{\mu+1}$ can be obtained by differentiating the equations of F_μ with respect to each variable in turn and eliminating first derivatives by means of the system (3.1). Likewise, it is clear that if $F_1, \cdots, F_\mu$ can be solved for $u_1, \cdots, u_p$ and if $F_{\mu+1}$ can be solved for $u_{p+1}, \cdots, u_q$ when $u_1, \cdots, u_p$ have been eliminated, the same solution is obtained by solving $F_1, \cdots, F_\mu, F_{\mu+1}$ for $u_1, \cdots, u_q$; and conversely. Hence Riquier's general process of reduction to passive form applied to (3.1) gives the theorem for compatibility of systems of total differential equations.[6]

4. **Additional results furnished by Riquier's methods.**[7] With each monomial m of a complete set M associate an arbitrarily chosen set X_m of the independent variables. The variables X_m will be called *differentiators* for m;

[6] O. Veblen and J. M. Thomas, Projective Invariants of Affine Geometry of Paths, these Annals, (2), vol. 27 (1926), p. 290. The theorem, in substance, was previously employed by J. E. Wright, Invariants of Quadratic Differential Forms, Cambridge, 1908.

[7] This section was added February 25, 1933.

the others, *non-differentiators*. If m has a non-differentiator x which is a multiplier, rewrite the corresponding term in (2.1) as

$$mf_m + mxf_{mx}, \tag{4.1}$$

so that x becomes a non-multiplier for m. Repeat this for all the monomials. Let the aggregate of the expressions like mxf_{mx} introduced in the process be called φ_1.

Multiply m in turn by each variable which is simultaneously differentiator and non-multiplier for it to obtain the set m_i. Write φ_1 so that each term of it which is divisible by m occurs in an expansion pf_p, where p is multiple of an m_i. If p/m involves only differentiators of m, adjoin that p to M and give it the differentiators of m. Perform the operation (4.1) for each p so added. Call the residual terms φ_2.

Repeat the process of the preceding paragraph for a second of the original monomials and with respect to φ_2. When all the monomials of the original set M have been so employed, we have an augmented set $\bar{M}$ with the following property: *the product of any monomial of M by an arbitrary monomial in its differentiators is the product of a monomial of $\bar{M}$ by a monomial in its multipliers; all of the latter are differentiators; and every monomial of $\bar{M}$ is multiple of a definite monomial of M by differentiators alone.* The final residue φ has no term which is the product of a monomial of M by differentiators alone.

Consider a system S which satisfies the hypotheses of Theorem 6 of the former paper and whose left members form sets $\bar{M}$ derived in the above manner from complete sets M.[8] Let it be proposed to find expansions satisfying the equations of S on the varieties defined by equating to zero the corresponding non-differentiators. If we augment each initial determination, as originally defined, by the terms in the corresponding φ and include in $S_a, S_{a+1}, \cdots, S_{A+1}$ only those equations whose left members correspond to multiples of $\bar{M}$ by differentiators, the demonstration in §4 of the previous paper holds verbatim. Theorem 6 is, therefore, valid when $S_a, S_{a+1}, \cdots, S_{A+1}$ are given the new interpretation. Since all non-differentiators are non-multipliers, the theorem furnishes an answer to the proposed problem. In order that an equation be satisfied for all values of a non-multiplier which is a differentiator, a corresponding integrability condition must be satisfied.

To select from the totality of systems satisfying the hypotheses of Theorem 6 a general type we need only to have in mind the satisfaction of conditions VII, p. 295 and IX, p. 297. This can be accomplished and the integrability conditions expressed by following the trail blazed by Riquier. The convergence is assured by Theorem 6.

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[8] If the set corresponding to any u is not complete, we complete it. If a monomial adjoined in so doing can only be obtained from the original set by multiplication with a non-differentiator, the corresponding expression (2.2) is put with the initial determination. Otherwise the corresponding equation is adjoined to S and assigned the same differentiators as the equation from which it arose.

T. Y. Thomas and Titt have considered a special case of the above problem. They let the set X comprise all the variables for some of the equations and the last $n - \alpha$ variables for all the rest. They satisfy IX by taking the corresponding p's zero on the variety.[9] This same condition helps to satisfy VII and facilitates the discussion of the integrability conditions. In addition, they prove the convergence and discuss at length sets of monomials.

5. **Concluding remarks.** Mr. J. F. Ritt has kindly pointed out to me that the proof given at the top of p. 288 of my previous paper is invalid because the statement "$i_2 \geqq j_2, \cdots, i_n \geqq j_n$" on line 4 is incorrect, and that the proof is easily accomplished by mathematical induction.

On page 289, second line from bottom, for $\partial^2 u/\partial y \partial z$ read $\partial^3 u/\partial y^2 \partial z$.

On page 290, line 3, insert "to" after "naturally."

On page 297, formula (20), for i's in second denominator read j's.

On page 298, line 6, for "coefficient" read "coefficients."

On page 301, formula (32), for ∂u^2 read $\partial^2 u$.

The type of determinant employed on page 293 has been studied by Bray[10] in another connection. He proves, essentially, Theorem 5.

Finally, it might not be amiss to stress the fact that the constants $P_{i\alpha j\beta}$ involved in (19) and in assumption IX are obtained by evaluating the coefficients $p_{i\alpha j\beta}$ in (17) for the x's equal to their initial values and the derivatives equal to the corresponding values, as is done in the example on p. 302. In other words, when Theorem 6 is applied, it is not necessary to make the initial values zero by carrying out transformation (16) in order to have the values of the P's.

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[9] Compare the remark immediately following the statement of IX, p. 297 of the former paper.

[10] H. E. Bray, *Rates of Exchange*, American Mathematical Monthly, vol. 29 (1922), p. 368.

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MOMENT-GENERATING OPERATORS FOR DETERMINANTS OF PRODUCT MOMENTS IN SAMPLES FROM A NORMAL SYSTEM

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I. Introduction

A considerable number of commonly used statistical functions, when generalized for samples from a normal multivariate population, can be expressed in terms of the determinants and their principal minors of matrices of second order moments in the sample. In a recent paper the author[2] considered generalizations of several statistical functions and found their moments from Wishart's[3] generalized product moment distribution by a method of variation of the sample number. Sampling distributions of such functions were found by solving the resulting moment equations, and the uniqueness of the solutions established by the theory of closure. In the present paper operational methods will be developed for finding the moments of a more general class of statistical functions directly from the probability law of the sample. The generality lies in the fact that some of the variates are treated as independent or fixed, that is, they are not subject to sampling variations. Hence the results will extend to least-square regression problems in which values of the independent variates are known without error. They are also applicable, as will be seen in the case of the multiple correlation coefficient, to problems in which an arbitrary number of the independent variates are subject to sampling variation.

We shall be primarily interested in deriving a method of obtaining general moment formulas by establishing the validity of certain operations on the population parameters involved in the probability law of the sample. However, several examples will be given in the section on Applications to show how the operators provide an effective analytical method of solving some of the more complicated sampling problems; e.g. that of the multiple correlation coefficient, some of the generalized Neyman-Pearson λ criteria, and the generalized "Student" ratio. All of the results of the author's paper cited above can be obtained at once by means of the operators.

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[1] International Research Fellow.

[2] Certain generalizations in the analysis of variance, Biometrika, Vol. XXIV (1932) pp. 471–494.

[3] J. Wishart, The generalized product moment distribution in samples from a normal multivariate population, Biometrika, Vol. XXA (1928) pp. 32–52. In this paper the distribution is established by geometrical argument. For an analytical proof of the same distribution see J. Wishart and M. S. Bartlett, The generalized product moment distribution in a normal system, Proceedings of the Cambridge Philosophical Society, Vol. 29 (1933) pp. 260–270.

In what is to follow, we shall be concerned only with the case of real variables. An important part will be played by a symmetric matrix of elements A_{rs} $(r, s = 1, 2, \cdots n)$ which will be denoted by $\| A_{rs} \|$. Let $\| A_{ij} \|$ be that part of $\| A_{rs} \|$ for which $(i, j = 1, 2, \cdots t)$, and $\| A_{pq} \|$ that for which $(p, q = t+1, t+2, \cdots n)$. The remaining elements of $\| A_{rs} \|$ will be of the form A_{ip}. The only condition to be imposed on $\| A_{rs} \|$ is that $\| A_{ij} \|$ be positive definite. Unless otherwise specified, i and j, when used as subscripts, are to take the values $1, 2, \cdots t$, and p and q the values $t+1, t+2, \cdots n$, while r and s are to take all values $1, 2, \cdots n$. When a quantity involving the subscripts i, j, p, q, r and s is preceded by Σ, summation will be understood with respect to all subscripts, otherwise the subscripts of summation will be indicated. All multiple integrals will be represented by a single integral sign, and when no limits of integration are given, the limits $-\infty$ to $+\infty$ will be understood for all variables of integration. The process of replacing a quantity Q by Q' will be denoted by writing $Q: \rightarrow Q'$. We shall make frequent use of Jacobi's theorem[4] in determinants that if D is the adjoint of any determinant d and m and M are corresponding k-rowed minors of d and D respectively, then M is equal to the product of d^{k-1} by the algebraic complement of m. We shall have several occasions to use the fact that if the symmetric matrix $\| c_{ij} \|$ $(i, j = 1, 2, \cdots t)$ is positive definite and $\{d_i\}$ any set of real numbers, then

$$\int e^{-\theta} d\bar{x} = \pi^{\frac{t}{2}} \mid c_{ij} \mid^{-\frac{1}{2}} e^{\varphi} \tag{1.1}$$

where $\theta = \sum_{i,j=1}^{t} c_{ij} x_i x_j + 2 \sum_{i=1}^{t} d_i x_i$, $\varphi = \sum_{i,j=1}^{t} c^{ij} d_i d_j$, $c^{ij} = \frac{\bar{c}_{ij}}{|c_{ij}|}$ $\bar{c}_{ij}$ is the cofactor of c_{ij} in $\mid c_{ij} \mid$ and the integral is taken over the entire t-dimensional space of the x's.

Most of the results will involve gamma functions which will, in practically all cases, appear in ratios of the form $\Gamma\left(\frac{a+b}{2}\right) \Big/ \Gamma\left(\frac{a}{2}\right)$. To simplify notation we shall denote this ratio by $\psi(a, b)$. Since we are considering only real variables the restrictions $a + b > 0$ and $a > 0$ will be made.

II. Moments of determinants of total product moments

1. **The problem.** We shall suppose that the simultaneous distribution in an infinite population of the variates x_i $(i = 1, 2, \cdots t)$ depends on an additional set of variates x_p $(p = t+1, t+2, \cdots n)$ in such a way that the probability law of the variates $\{x_i\}$ is given by

$$f(\bar{x}_i) = Ke^{-\Sigma A_{rs} x_r x_s} \tag{2.1}$$

[4] Cf. H. W. Turnbull, The theory of determinants, matrices and invariants, (Blackie and Son, London 1926) p. 77.

where $A_{rs} \equiv A_{sr}$ and $||A_{ij}||$ is positive definite, $\bar{x}_i$ is the set of variates x_i ($i = 1, 2, \cdots t$) and

$$1/K = \int e^{-\Sigma A_{rs}x_r x_s}\, d\bar{x}_i \tag{2.2}$$

$d\bar{x}_i$ being the product of the differentials of the x's in $\{\bar{x}_i\}$. The sets $\{x_i\}$ and $\{x_p\}$ will be referred to as dependent and independent variates respectively.

It is clear that $e^{-\Sigma A_{pq}x_p x_q}$ is a constant as far as the probability function of the x_i is concerned and could be included in K, but to do so would remove a part of the matrix $||A_{rs}||$ and it is essential at present that this matrix remain complete.

The integral in (2.2) can be written as

$$\int e^{-\Sigma A_{ij}x_i x_j - 2\Sigma A_{ip}x_i x_p - \Sigma A_{pq}x_p x_q}\, d\bar{x}_i, \tag{2.3}$$

and it follows from (1.1) that the value of (2.3) is

$$\pi^{\frac{t}{2}} |A_{ij}|^{-\frac{1}{2}} e^{-\theta} \tag{2.4}$$

where

$$\theta = \Sigma A_{pq}x_p x_q - \Sigma A_{ip}A_{qj}\frac{\bar{A}_{ij}}{|A_{ij}|}x_p x_q.$$

There will be no occasion to use the determinant of $||A_{rs}||$; only the determinant of $||A_{ij}||$ and its minors will arise. Therefore, without ambiguity, we can denote $|A_{ij}|$ by A and let $A^{ij} = \bar{A}_{ij}/|A_{ij}|$, where $\bar{A}_{ij}$ is the cofactor of A_{ij} in $|A_{ij}|$, that is, A^{ij} is the element in the i-th row and j-th column of the inverse of $||A_{ij}||$. Thus if we set $B_{pq} = A_{pq} - \sum_{i,j} A_{ip}A_{qj}A^{ij}$ we can write

$$K = \pi^{-\frac{t}{2}} A^{\frac{1}{2}} e^{\Sigma B_{pq}x_p x_q}. \tag{2.5}$$

Now let a sample of N individuals be drawn from the population distributed according to (2.1). By drawing an individual from (2.1) we mean the selection of a set of values of the dependent variates in the probability sense for an arbitrarily chosen set of values of the independent variates. In the α-th drawing let x_p have the assigned value $x_{p\alpha}$ ($p = t+1, t+2, \cdots n$) and let the value of x_i drawn be $x_{i\alpha}$ ($i = 1, 2, \cdots t$). Since K depends on $\{x_p\}$ let its value belonging to the α-th drawing be K_α. Then the probability law of the sample formed when α takes the values $1, 2, \cdots N$ is

$$\prod_{\alpha=1}^{N} f(\bar{x}_{i\alpha}) = \bar{K}\, e^{-\Sigma A_{rs}x_{r\alpha}x_{s\alpha}} \tag{2.6}$$

where $\bar{K} = \prod_{\alpha=1}^{N} K_\alpha$ and $\Sigma A_{rs}x_{r\alpha}x_{s\alpha}$ is summed over all values of r, s and α.

By writing $\sum_{\alpha=1}^{N} x_{r\alpha}x_{s\alpha} = a_{rs}$ and using the fact that the integral of (2.6)

from $-\infty$ to $+\infty$ with respect to the $x_{i\alpha}$ ($i = 1, 2, \cdots t; \alpha = 1, 2, \cdots N$) is unity, it follows at once that

$$\int e^{-\Sigma A_{rs}a_{rs}}\, d\bar{x}_{i\alpha} = \pi^{\frac{Nt}{2}} A^{-\frac{N}{2}} e^{-\Sigma B_{pq}a_{pq}}. \tag{2.7}$$

The quantity a_{rs} may be called the *total* product moment of the r-th and s-th variates.

We shall consider the relationship (2.7) as a basis for obtaining the sampling moments, when they exist, of various functions of determinants similar in structure to $| a_{rs} |$ and its principal minors. In general, if $F(\bar{a}_{rs})$ is a function of the form $\prod_{\beta=1}^{m} | a_{\mu_\beta \nu_\beta} |^{k_\beta}$ where the set of determinants is selected from $| a_{rs} |$ and its principal minors, the problem of this section is to find the value of

$$E(F) = \int P(\bar{a}_{rs}) F(\bar{a}_{rs})\, d\bar{x}_{i\alpha} \tag{2.8}$$

for all values of the h_β for which $E(F)$ exists, where

$$P(\bar{a}_{rs}) = e^{-\Sigma A_{rs}a_{rs}}.$$

We shall find the value of the integral defining $E(F)$ by performing certain operations on the A_{rs} which are commutative with the integral signs and otherwise valid for both sides of (2.7) which will yield $E(F)$ on the left of (2.7) and which will be equal to the resulting quantity on the right.

2. **Evaluation of a definite integral.** It will be noticed that (2.7) holds for all values of the A_{rs} for which $|| A_{ij} ||$ is positive definite. Now if $Q = \Sigma A_{ij} y_i y_j$ is the positive definite quadratic form associated with $|| A_{ij} ||$, it follows at once that

$$Q' = \sum_{i,j} \left(A_{ij} + \sum_{g=1}^{k} \xi_{ig}\xi_{jg} \right) y_i y_j \tag{2.9}$$

is also positive definite for all values of k and the ξ's, because Q' can be written as $Q + \sum_{g=1}^{k} \left(\sum_{i=1}^{t} \xi_{ig} y_i \right)^2$. Thus, the matrix $\left\| A_{ij} + \sum_{g=1}^{k} \xi_{ig}\xi_{jg} \right\|$ is positive definite and hence (2.7) holds when the substitution

$$A_{rs} :\rightarrow A'_{rs} \qquad (r, s = 1, 2, \cdots n) \tag{2.10}$$

is made, where $A'_{rs} = A_{rs} + \sum_{g=1}^{k} \xi_{rg}\xi_{sg}$. The effect of (2.10) on the left side of (2.7) is to subtract in the exponent of the function $P(\bar{a}_{rs})$ k quadratic forms in the ξ's, each of which has the matrix $|| a_{rs} ||$. Thus, we shall refer to the ξ's introduced by (2.10) as a k-fold set of *ξ-parameters* associated with $|| a_{rs} ||$. If

$|| A_{\mu\nu} ||$ is any principal minor of any order of $|| A_{rs} ||$, and if in the A'_{μ}, the further substitution

$$A_{\mu\nu} : \rightarrow A_{\mu\nu} + \sum_{\varrho=1}^{k_1} \xi'_{\mu\varrho}\, \xi'_{\nu\varrho} \tag{2.11}$$

is made in (2.7) then the ξ''s will be ξ-parameters associated only with the minor $|| a_{\mu\nu} ||$. Clearly (2.10) and (2.11) are commutative and the result of the two substitutions is

$$A_{rs} : \rightarrow A_{rs} + \sum_{\varrho=1}^{k} \xi_{r\varrho}\, \xi_{s\varrho} + \gamma(\mu, \nu)_{rs} \sum_{\varrho=1}^{k_1} \xi'_{r\varrho}\, \xi'_{s\varrho} \tag{2.12}$$

where $\gamma(\mu, \nu)_{rs}$ is unity when r and s take values within the minor specified by μ and ν and zero otherwise. In general we can introduce into (2.7) a set of ξ-parameters associated with each of an arbitrary set of principal minors $|| a_{\mu_h\nu_h} ||$ $(h = 1, 2, \cdots m)$ and $|| a_{rs} ||$ by making the substitution

$$A_{rs} : \rightarrow A_{rs} + \sum_{\varrho=1}^{k} \xi_{r\varrho}\, \xi_{s\varrho} + \sum_{h=1}^{m} \left[\gamma(\mu_h, \nu_h)_{rs} \sum_{\varrho=1}^{k_h} \xi^{(h)}_{r\varrho}\, \xi^{(h)}_{s\varrho} \right]. \tag{2.13}$$

The right side of (2.13) can be identified without difficulty with an expression of the form $A_{rs} + \sum_{\varrho=1}^{N} \eta_{r\varrho}\eta_{s\varrho}$ $\left(N = k + \sum_{h=1}^{m} k_h\right)$ where some of the η's will be zero, depending on the set of minors selected. Therefore, as in (2.9) $|| A_{ij} ||$ will remain positive definite when (2.13) is applied. Some of the elements of the matrix $|| A_{rs} ||$ may be zero, but such a case will present no difficulty if we regard them as non-zero until after the substitution (2.13).

Now let us consider (2.7) after applying (2.10) with a l-fold set of ξ-parameters for $|| a_{rs} ||$. We get

$$\int P'(\bar{a}_{rs})\, d\bar{x}_{i\alpha} = \pi^{\frac{Nl}{2}} A'^{-\frac{N}{2}}\, e^{-\Sigma B'_{pq}\, a_{pq}} \tag{2.14}$$

where $A' = |\, A_{ij} + \xi_i\xi_j \,|$, and B'_{pq} and $P'(\bar{a}_{rs})$ are the values of B_{pq} and $P(\bar{a}_{rs})$ after applying the set of substitutions $A_{rs} : \rightarrow A_{rs} + \xi_r\xi_s$, $(r, s = 1, 2, \cdots n)$.

It will be noticed that the integrand of the expression on the left in (2.14) is: (1) non-negative for all values of the ξ's and x's from $-\infty$ to $+\infty$ and (2) continuous and therefore summable in every finite rectangular cell in the space of the ξ's and x's. Therefore[5] if the integral is regarded as a repeated integral and exists for one order of integration, it exists for all orders and has the same value. Now, by integrating the left side of (2.14) with respect to the ξ's from $-\infty$ to $+\infty$, we find at once by (1.1)

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[5] Cf. E. W. Hobson, The theory of functions of a real variable, Vol. II Second Edition (1926) p. 344.

$$\pi^{\frac{n}{2}} \int P(\bar{a}_{rs}) \mid a_{rs} \mid^{-\frac{1}{2}} d\bar{x}_{i\alpha} \tag{2.15}$$

which, when multiplied by $\bar{K}\pi^{-\frac{n}{2}}$, is the expression defining $E(\mid a_{rs} \mid^{-\frac{1}{2}})$.

The value of (2.15), if it exists, will be given by the integral of the right side of (2.14) with respect to the ξ's, that is, by

$$\pi^{\frac{Nt}{2}} \int A'^{-\frac{N}{2}} e^{-\Sigma B'_{pq} a_{pq}} d\bar{\xi}_r. \tag{2.16}$$

To evaluate this integral we have

$$B'_{pq} = (A_{pq} + \xi_p \xi_q) - \sum_{i,j} (A_{iq} + \xi_i \xi_q)(A_{pj} + \xi_p \xi_j) A'^{ij} \tag{2.17}$$

where A'^{ij} is the value of A^{ij} after applying (2.10) with a l-fold set of ξ's. It can be readily shown that

$$\Sigma \xi_i \xi_j A'^{ij} = 1 - A/A', \tag{2.18}$$

and therefore

$$B'_{pq} = A_{pq} - \sum_{i,j} A_{iq} A_{pj} A'^{ij} + \frac{A}{A'} \xi_p \xi_q - 2 \sum_{i,j} A_{iq} A'^{ij} \xi_j \xi_p. \tag{2.19}$$

We shall suppose that the values $\{x_{p\alpha}\}$ are so chosen that $\| a_{pq} \|$ is positive definite. This will be true when no such relation of the form $\sum_p c_p x_{p\alpha} = 0$ exists for all values of α, where the c's are not all zero. Under these conditions (2.16) can be integrated with respect to the ξ_p $(p = t+1, t+2, \cdots n)$. The integral, as far as the ξ_p are concerned is clearly of the form of (1.1). Therefore we have for (2.16)

$$\pi^{\frac{Nt+n-t}{2}} A^{-\frac{n-t}{2}} a^{-\frac{1}{2}} e^{-\Sigma A_{pq} a_{pq}} \int A'^{-\frac{N-n+t}{2}} e^{Q_1 - Q_2} d\bar{\xi}_i, \tag{2.20}$$

where $a = \mid a_{pq} \mid$, $Q_1 = \Sigma A_{ip} A_{qj} A'^{ij} a_{pq}$ and

$$Q_2 = -A'/A \, \Sigma A_{iq} A'^{ij} A_{gk} A'^{gh} a_{pq} a_{kl} a^{pl} \xi_j \xi_h,$$

a^{pl} being the element in the p-th row and l-th column of the inverse of $\| a_{pq} \|$, and the summation in Q_2 extending over all suffixes where $g, h = 1, 2, \cdots t$ and $k, l = t+1, t+2, \cdots n$. But $\sum_l a_{kl} a^{pl}$ is unity if $k = p$, and zero if $k \neq p$.

Furthermore, it can be shown that

$$\sum_{i,j} A_{iq} A'^{ij} \xi_j = \frac{A}{A'} \sum_{i,j} A_{iq} A^{ij} \xi_j. \tag{2.21}$$

Therefore

$$Q_2 = -A/A' \, \Sigma A_{iq} A^{ij} A_{gp} A^{gh} a_{pq} \xi_j \xi_h. \tag{2.22}$$

Now we can write Q_1 in terms of bordered determinants as

$$Q_1 = 1/A' \Sigma D_{pq} a_{pq} + A/A' \Sigma A_{iq} A_{pj} A^{ij} a_{pq}, \tag{2.23}$$

where

$$D_{pq} = \begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1t} & A_{1q} & \xi_1 \\ A_{21} & A_{22} & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ A_{t1} & \cdots & \cdots & A_{tt} & A_{tq} & \xi_t \\ A_{p1} & \cdots & \cdots & A_{pt} & 0 & 0 \\ \xi_1 & \cdots & \cdots & \xi_t & 0 & 0 \end{vmatrix}.$$

Expanding D_{pq} by the border method and using Jacobi's theorem on the minors of A of order $t - 2$ which arise, we can write

$$Q_1 = A/A' \left(\Sigma A_{pj} A_{iq} A^{ij} A^{gh} a_{pq} \xi_g \xi_h + \Sigma A_{iq} A_{pj} A^{ij} a_{pq} - \Sigma A_{iq} A_{gp} A^{ij} A^{gh} a_{pq} \xi_j \xi_h\right). \tag{2.24}$$

But similar to (2.18) we have

$$\Sigma A^{gh} \xi_g \xi_h = -1 + A'/A. \tag{2.25}$$

Making this simplification in (2.24) and using (2.22) we finally have

$$Q_1 - Q_2 = \Sigma A_{iq} A_{pj} A^{ij} a_{pq}$$

which is independent of the ξ_i. Therefore (2.20) can be written as

$$\pi^{\frac{Nt+n-t}{2}} A^{-\frac{n-t}{2}} a^{-\frac{1}{2}} e^{-\Sigma B_{pq} a_{pq}} \int A'^{-\frac{N-n+t}{2}} d\bar{\xi}_i. \tag{2.26}$$

Now consider the integral with respect to the ξ_i, and introduce the value of A' from (2.25) into the integrand. The quadratic $\Sigma A^{ij} \xi_i \xi_j$ is positive definite and hence can be reduced to a sum of squares with coefficients of unity by a non-singular transformation

$$\xi_j = \sum_i c_{ij} y_i \qquad (j = 1, 2, \cdots t) \tag{2.27}$$

so that $|c_{ij}| = A^{\frac{1}{2}}$ and

$$\int A'^{-\frac{N-n+t}{2}} d\bar{\xi}_i = A^{-\frac{N-1-n+t}{2}} \int (1 + \Sigma y_i^2)^{-\frac{N-n+t}{2}} d\bar{y}_i. \tag{2.28}$$

Because of the symmetry of the integrand we can write

$$\int (1 + \Sigma y_i^2)^{-\frac{N-n+t}{2}} dy_i = 2^t \int_0^\infty \cdots \int_0^\infty (1 + \Sigma y_i^2)^{-\frac{N-n+t}{2}} dy_i \tag{2.29}$$

Making the transformation

$$y_i^2 = \theta_i (1 + y_1^2 + \cdots + y_{i-1}^2), \qquad (i = 1, 2, \cdots t) \tag{2.30}$$

we can express the right side of (2.29) in terms of gamma functions. Accordingly, we have

$$\int (1 + \Sigma y_i^2)^{-\frac{N-n+t}{2}} d\bar{y}_i = \Gamma^t(\tfrac{1}{2}) \prod_{i=1}^{t} \Psi(N - n + t + 1 - i, -1) \tag{2.31}$$

where the Ψ function has already been defined. The value of (2.16) is finally found to be

$$\pi^{\frac{Nt+n}{2}} \prod_{i=1}^{t} [\Psi(N - n + t + 1 - i, -1)]\, A^{-\frac{N-1}{2}}\, a^{-\frac{1}{2}}\, e^{-\Sigma B_{pq} a_{pq}}, \tag{2.32}$$

which, together with all previous assumptions, will clearly exist when $N > n$.

The problem involving the expressions (2.14) to (2.32) may be conveniently summarized in the following

THEOREM I: *If* (a) $|| A_{ij} ||$ *is a positive definite matrix,* (b) *the values* $\{x_{p\alpha}\}$ *are assigned so that* $|| a_{pq} ||$ *is positive definite and* (c) $N > n$, *then the integrals* (2.15) *and* (2.16) *exist and are equal and their value is given by* (2.32).

3. **The moment-generating operators.** Since (2.7) is valid for all values of the A_{rs} for which $|| A_{ij} ||$ is positive definite, it will hold when the substitution

$$A_{rs} :\rightarrow A_{rs\beta} \equiv A_{rs} + \beta_{rs}, \qquad (\beta_{rs} \equiv \beta_{sr};\ r, s = 1, 2, \cdots n), \tag{2.33}$$

is made, provided the β_{ij} are sufficiently small. Making this substitution and multiplying by $\bar{K}$, we get

$$\bar{K} \int P_\beta(\bar{a}_{rs})\, d\bar{x}_{i\alpha} = \bar{K} G_\beta \tag{2.34}$$

where $G_\beta = \pi^{\frac{Nt}{2}} A_\beta^{-\frac{N}{2}} e^{-\Sigma B_{pq\beta} a_{pq}}$, $A_\beta = | A_{ij\beta} |$ and $B_{pq\beta}$ and $P_\beta(\bar{a}_{rs})$ are the expressions for B_{pq} and $P(\bar{a}_{rs})$ after applying the substitution (2.33). If we let $e^{-\Sigma \beta_{rs} a_{rs}} = g_\beta$ then the left side of (2.34) is clearly the expression defining $E(g_\beta)$.

a. **The operator $I_{\mu\nu}$.** Let $|| a_{\mu\nu} ||$ be a principal minor of $|| a_{rs} ||$, and let $I_{\mu\nu}$ denote the operation of applying the substitution

$$\beta_{\mu\nu} :\rightarrow \beta_{\mu\nu} + \xi_\mu \xi_\nu \tag{2.35}$$

to (2.34) and integrating with respect to the ξ's from $-\infty$ to $+\infty$, and then multiplying by $\pi^{-\frac{m}{2}}$ where μ and ν are to take on the values specified by $|| a_{\mu\nu} ||$ and m is the order of $|| a_{\mu\nu} ||$.

Let us consider I_{rs}. It follows from Theorem I that I_{rs} is commutative with

the integrals in (2.34) and that the application of I_{rs} to the left side of (2.34) yields

$$\bar{K} \int P_\beta(\bar{a}_{rs}) \mid a_{rs} \mid^{-\frac{1}{2}} d\bar{x}_{ia}, \tag{2.36}$$

which is the integral defining $E(g_\beta \mid a_{rs} \mid^{-\frac{1}{2}})$, while the I_{rs} of the right side of (2.34) is

$$\bar{K} \prod_{i=1}^{t} [\Psi(N - n + t + 1 - i, -1)]\, a^{-\frac{1}{2}} A_\beta^{\frac{1}{2}} G_\beta. \tag{2.37}$$

Consider the effect of operating on (2.34) with I_{rs} λ successive times. Denoting the λ-fold operation by I_{rs}^{λ}, we have at once by λ successive applications of Theorem I that

$$E(g_\beta \mid a_{rs} \mid^{-\frac{\lambda}{2}}) = I_{rs}^{\lambda} \bar{K} G_\beta = \bar{K} \prod_{i=1}^{t} [\Psi(N - n + t + 1 - i, -\lambda)]\, a^{-\frac{\lambda}{2}} A_\beta^{\frac{\lambda}{2}} G_\beta \tag{2.38}$$

provided $N - n + 1 - \lambda > 0$. The $-\frac{\lambda}{2}$-th moment of $\mid a_{rs} \mid$ can be found from (2.38) by setting all of the β's equal to zero. However, the β's are to be set equal to zero after performing the operations $I_{rs}^{\lambda} \bar{K} G_\beta$. We observe that the product $\bar{K} G_\beta$ is unity when the β's are zero.

To return to the more general case, suppose $\| a_{\mu\nu} \|$ is any principal minor of $\| a_{rs} \|$ which includes $\| a_{pq} \|$, that is, such that $\mu, \nu = k + 1, k + 2, \cdots n$ $(0 \leq k < t)$ and apply $I_{\mu\nu}$ as defined above to (2.34). On the left we get $E(g_\beta \mid a_{\mu\nu} \mid^{-\frac{1}{2}})$, and on the right $I_{\mu\nu} \bar{K} G_\beta$ which has the value

$$\pi^{\frac{Nt}{2}} A_\beta^{-\frac{n-t}{2}}\, a^{-\frac{1}{2}}\, e^{-\Sigma B_{pq\beta} a_{pq}} \int A_\beta'^{-\frac{N-n+t}{2}}\, d\bar{\xi}_g \tag{2.39}$$

where the integral is $(t - k)$-fold with respect to ξ_g $(g = k + 1, k + 2, \cdots t)$ and A_β' is the determinant $\mid A_{ij\beta} \mid$ after applying the substitution β_{gh} : $\rightarrow \beta_{gh} + \xi_g \xi_h$ $(g, h = k + 1, k + 2, \cdots t)$. By straight forward integration of (2.39) we finally obtain the following result of applying $I_{\mu\nu}$ to (2.34),

$$E(g_\beta \mid a_{\mu\nu} \mid^{-\frac{1}{2}}) = I_{\mu\nu} \bar{K} G_\beta = \bar{K} \prod_{i=1}^{t-k} [\Psi(N - n + t + 1 - i, -1)]\, A_\beta^{\frac{1}{2}} \mid A_{bc\beta} \mid^{-\frac{1}{2}} a^{-\frac{1}{2}} G_\beta \tag{2.40}$$

where $b, c = 1, 2, \cdots k$. The right side holds for $N - n + k > 0$, which insures the validity of the interchange of $I_{\mu\nu}$ and the integral sign on the left in (2.34).

For the case of repeated operations with $I_{\mu\nu}$ we find similar to (2.38), the result

$$E\left(g_\beta \mid a_{\mu\nu} \mid^{-\frac{\lambda}{2}}\right) = I_{\mu\nu}^{\lambda} \bar{K} G_\beta = \bar{K} \prod_{i=1}^{t-k} [\Psi(N - n + t + 1 - i, -\lambda)]\, A_\beta^{\frac{\lambda}{2}} \mid A_{bc\beta} \mid^{-\frac{\lambda}{2}} a^{-\frac{\lambda}{2}} G_\beta \tag{2.41}$$

which is valid when $N - n + k + 1 - \lambda > 0$, and reduces to (2.38) when $k = 0$. The $-\frac{\lambda}{2}$-th moment of $|\, a_{\mu\nu} \,|$ is found by setting the β's equal to zero.

b. **The inverse operator $I^{-1}_{\mu\nu}$.** Thus far we have considered only a method of finding moments of $|\, a_{rs} \,|$ and its principal minors which include $|\, a_{pq} \,|$, when the orders of the moments are multiples of $-\frac{1}{2}$. In order to find the positive moments of $|\, a_{\mu\nu} \,|$ of order $\frac{\lambda}{2}$ ($\lambda = 1, 2, 3, \cdots$) we shall make use of the following

THEOREM II: *Let (a) and (b) of Theorem I be assumed and $N - n + k + 1 > 0$. If ρ is a positive integer and M_ρ has the value of the last expression in* (2.41) *with $\lambda = -\rho$, then*

$$\bar{K} \int P_\beta(\bar{a}_{rs}) \,|\, a_{\mu\nu} \,|^{\rho} \, d\bar{x}_{i\alpha} = M_\rho. \tag{2.42}$$

We shall prove[6] the theorem by making use of certain invariant properties of $P_\beta(\bar{a}_{rs})$ and $|\, a_{\mu\nu} \,|$ under linear transformations of the x's. It is sufficient to consider a series of two transformations. First, let T_1 be a transformation of the form

$$x_{r\alpha} = \sum_{s=1}^{n} g_{rs} \, x'_{s\alpha} \qquad (r = 1, 2, \cdots n; \alpha = 1, 2, \cdots N) \tag{2.43}$$

where, for $r > k$, $g_{rr} = 1$, $g_{rs} = 0$ ($r \neq s$), and the matrix $\| g_{rs} \|$ has the property that $|\, g_{bc} \,|^2 \cdot |\, A_{bc\beta} \,| = 1$ ($b, c = 1, 2, \cdots k$) while the g_{rs} ($1 \leqq r \leqq k < s \leqq n$) are functions of the A_{rs} which need not be given, and furthermore, T_1 is to be such that its application to $P_\beta(\bar{a}_{rs})$ will yield

$$e^{-\sum_{c,\alpha} x'^2_{c\alpha}} \, P_{o\beta}(\bar{a}'_{\mu\nu}). \tag{2.44}$$

where $P_{o\beta}(\bar{a}'_{\mu\nu}) = e^{-\sum_{\mu,\nu=k+1}^{n} \bar{A}_{\mu\nu} a'_{\mu\nu}}$. It follows from (2.34) that the $\bar{A}_{\mu\nu}$ must be such that $|\, \bar{A}_{lm} \,| \cdot |\, A_{bc\beta} \,| = A_\beta$ ($l, m = k + 1, k + 2, \cdots t$). The existence of a matrix $\| g_{rs} \|$ satisfying these conditions is evident from the fact that it can be regarded as the matrix of the result of two transformations T'_1 and T''_2, where T'_1 is of the form

$$x_{r\alpha} = \sum_{s=1}^{n} g'_{rs} \, z_{s\alpha} \qquad (r = 1, 2, \cdots n, \alpha = 1, 2, \cdots N) \tag{2.45}$$

where $g'_{rs} = 1$ ($r = s$), $g'_{rs} = g_{rs}$ ($1 \leqq r \leqq k < s \leqq n$) and zero otherwise, which will reduce the quadratic form $\Sigma A_{rs} x_{r\alpha} x_{s\alpha}$, for each value of α, to a form

[6] This proof is an extension of the argument suggested by Professor J. von Neumann for the special case when $k = 0$ and $t = n$.

having a matrix $\| A'_{rs} \|$, where $A'_{rs} \equiv A_{rs}$ $(r, s = 1, 2, \cdots k)$, $A'_{\mu\nu} \equiv \bar{A}_{\mu\nu}$ $(\mu, \nu = k+1, \cdots n)$ and zero otherwise. T''_1 is of the form

$$z_{r\alpha} = \sum_{s=1}^{n} g''_{rs} x'_{s\alpha} \tag{2.46}$$

where $g''_{rs} = g_{rs}$ $(r, s = 1, 2, \cdots k)$, $g''_{rr} = 1$ $(r = k+1, k+2, \cdots n)$, and zero otherwise, such that it will reduce $\sum_{b,c,\alpha} A_{bc\beta} x_{b\alpha} x_{c\alpha}$ to a sum of Nk squares having coefficients of unity. Clearly, $\| g'_{rs} \|$ and $\| g''_{rs} \|$ are non-singular, and $\| g'_{rs} \| \cdot \| g''_{rs} \| = \| g_{rs} \|$.

Now we consider a second transformation T_2

$$x'_{\mu\alpha} = \sum_{\nu=k+1}^{n} h_{\mu\nu} x''_{\nu\alpha} \quad (\mu = k+1, k+2, \cdots n; \alpha = 1, 2, \cdots N) \tag{2.47}$$

where $h_{\mu\mu} = 1$ $(\mu > t)$, $h_{\mu\nu} = 0$ $(\mu \neq \nu, \mu > t, \nu = k+1, k+2, \cdots n)$, and $\| h_{\mu\nu} \|$ has the property that $| h_{lm} |^2 \cdot | A_{lm} | = 1$ $(l, m = k+1, k+2, \cdots t)$, while the $h_{\mu\nu}$ $(k < \mu \leqq t < \nu \leqq n)$ are functions of the A_{rs} which need not be given. Furthermore, T_2 is to be such that its application to $P_{o\beta}(\bar{a}'_{\mu\nu})$ gives

$$e^{-\sum_{l,\alpha} x''^2_{l\alpha}} \cdot e^{-\sum \bar{B}_{pq} a''_{pq}} \tag{2.48}$$

where the $\bar{B}_{pq}$ are to be determined. The structure of T_2 is similar to that of T_1 and therefore transformations of type T_2 clearly exist satisfying the restrictions that have been imposed on $\| h_{\mu\nu} \|$. Applying T_2 to $| a'_{\mu\nu} |$ we get at once

$$| h_{\mu\nu} |^2 \, | a''_{\mu\nu} | \tag{2.49}$$

where $a''_{\mu\nu} = \sum_{\alpha=1}^{N} x''_{\mu\alpha} x''_{\nu\alpha}$. It will be noticed that $a''_{pq} \equiv a_{pq}$ $(p, q = t+1, t+2, \cdots n)$.

Since the Jacobian of the product of T_1 and T_2 for each value of α is $| h_{\mu\nu} | \cdot | g_{rs} | = A_\beta^{-\frac{1}{2}}$, we find that the application of T_1 and T_2 to the left side of (2.42) gives

$$\bar{K} \int P_\beta(\bar{a}_{rs}) \, | a_{\mu\nu} |^{\frac{\rho}{2}} \, d\bar{x}_{i\alpha} = \tag{2.50}$$

$$\bar{K} \, e^{-\Sigma \bar{B}_{pq} a_{pq}} \, A_\beta^{-\frac{N+\rho}{2}} \, | A_{bc\beta} |^{\frac{\rho}{2}} \, H$$

where

$$H = \int e^{-\sum_{i,\alpha} x''^2_{i\alpha}} \, | a''_{\mu\nu} |^{\frac{\rho}{2}} \, d\bar{x}''_{i\alpha}.$$

By setting $\rho = 0$ it follows from (2.34) that $\bar{B}_{pq} \equiv B_{pq\beta}$ $(p, q =$

$t+1, t+2, \cdots n$). Since H is independent of the A_{rs}, we can determine its value by operating on (2.50) with $I_{\mu\nu}^{\frac{\rho}{2}}$. Applying $I_{\mu\nu}^{\frac{\rho}{2}}$ and using (2.46), we find

$$\bar{K} \int P_\beta(\bar{a}_{rs})\, d\bar{x}_{i\alpha} = \bar{K} A_\beta^{-\frac{N}{2}} e^{-\Sigma B_{pq\beta} a_{pq}} \times a^{-\frac{\rho}{2}} \prod_{i=1}^{t-k} [\Psi(N-n+t+1+\rho-i, -\rho)]\, H \tag{2.51}$$

which exists if $N - n + k + 1 > 0$. The value of the expression on the left is given by (2.34). Hence

$$H = \pi^{\frac{Nt}{2}} a^{\frac{\rho}{2}} \prod_{i=1}^{t-k} [\Psi(N-n+t+1-i, \rho)]. \tag{2.52}$$

Therefore, (2.50) becomes

$$\bar{K} \int P_\beta(\bar{a}_{rs}) \mid a_{\mu\nu} \mid^{\frac{\rho}{2}} d\bar{x}_{i\alpha} = \bar{K} \prod_{i=1}^{t-k} [\Psi(N-n+t+1-i, \rho)] \times A_\beta^{-\frac{\rho}{2}} \mid A_{bc\beta} \mid^{\frac{\rho}{2}} a^{\frac{\rho}{2}} G_\beta \tag{2.53}$$

which completes the proof.

For the sake of convenience we shall let $I_{\mu\nu}^{-1}$ be an operator such that

$$I_{\mu\nu}^{-1} \bar{K} \int P_\beta(\bar{a}_{rs})\, d\bar{x}_{i\alpha} = \bar{K} \int P_\beta(\bar{a}_{rs}) \mid a_{\mu\nu} \mid^{\frac{1}{2}} d\bar{x}_{i\alpha} = E(g_\beta \mid a_{\mu\nu} \mid^{\frac{1}{2}}). \tag{2.54}$$

Then from Theorem II it follows that the application of $I_{\mu\nu}^{-1}$ to (2.34) yields

$$E(g_\beta \mid a_{\mu\nu} \mid^{\frac{1}{2}}) = M_1 = I_{\mu\nu}^{-1} \bar{K} G_\beta.$$

Corresponding to the expression (2.41) we have, by a ρ-fold application of $I_{\mu\nu}^{-1}$, the formula

$$E(g_\beta \mid a_{\mu\nu} \mid^{\frac{\rho}{2}}) = I_{\mu\nu}^{-\rho} \bar{K} G_\beta. \tag{2.55}$$

where $I_{\mu\nu}^{-\rho} \bar{K} G_\beta$ is understood to have the value M_ρ. This is a symbolic way of writing (2.53).

We have therefore shown that when λ is a positive or negative integer, the λ-th moment of $\mid a_{\mu\nu} \mid^{\frac{1}{2}}$ exists if both $N - n + k + 1$ and $N - n + k + 1 + \lambda$ are positive, and is given by (2.41) when the β's are zero.

4. **Simultaneous moments.** Before dealing with the general problem expressed by (2.8) let us consider the case of finding the value of $E(\mid a_{rs} \mid^h \mid a_{\mu\nu} \mid^\rho)$. Applying I_{rs}^{-2h} to (2.34), we get

$$E(g_\beta \mid a_{rs} \mid^h) = \bar{K} \prod_{i=1}^{t} [\Psi(N-n+t+1-i, 2h)]\, A_\beta^{-h} a^h G_\beta. \tag{2.56}$$

When g is a multiple of $\frac{1}{2}$ the application of $I_{\mu\nu}^{2g}$ to $(I_{rs}^{-2h}G_\beta)$ is simply to perform $2g$ times the operation of introducing into $(I_{rs}^{-2h}G_\beta)$ a 1-fold set of ξ-parameters associated with $|| a_{\mu\nu} ||$, integrating with respect to the ξ's and multiplying by $\pi^{-\frac{n-k}{2}}$. This compound operation will be denoted by

$$I_{\mu\nu}^{2g} I_{rs}^{-2h} G_\beta. \tag{2.57}$$

It can be shown, as in the case of a single operator, that the application of $I_{\mu\nu}^{-2g}$ to (2.56) yields

$$E(g_\beta \mid a_{rs} \mid^h \mid a_{\mu\nu} \mid^g) = \bar{K} \prod_{i=1}^{t} [\Psi(N - n + t + 1 - i, 2h)] \tag{2.58}$$

$$\times \prod_{i=1}^{t-k} [\Psi(N - n + t + 1 - i + 2h, 2g)] \times A_\beta^{-h-g} \mid A_{bc\beta} \mid^g a^{h+g} G_\beta,$$

and the result holds for all values of g and h for which condition (P) is satisfied in all of the Ψ's, where (P) is the condition that $a + b > 0$ and $a > 0$ in $\Psi(a, b)$. To find the simultaneous moment $E(\mid a_{rs} \mid^h \mid a_{\mu\nu} \mid^g)$ we set the β's $= 0$.

There is no difficulty involved in the above process if the operators are applied in the order just given. However, if the reverse order or any mixed order (e.g. $I_{rs}^{-2h_1} I_{\mu\nu}^{-2g_1} I_{rs}^{-2h_2} I_{\mu\nu}^{-2g_2} G_\beta$ where $h_1 + h_2 = h$ and $g_1 + g_2 = g$) is used, several troublesome features may be encountered. Firstly, $I_{\mu\nu}^{-2g} G_\beta$ will not exist if $N - n + k + 1 + 2g \leq 0$, whereas, under such conditions (2.54) will exist for certain values of h, and hence the further operation I_{rs}^{-2h} will have no meaning. Secondly, if $I_{\mu\nu}^{-2g} G_\beta$ does exist, then unless h is a multiple of $-\frac{1}{2}$ the meaning and validity of the operation I_{rs}^{-2h} $(I_{\mu\nu}^{-2g} G_\beta)$ would have to be treated by a procedure similar to that used in considering $I_{\mu\nu}^{\lambda} G_\beta$ when λ is a negative integer. A detailed treatment of the question of commutability of these operations, especially in the general case of m principal minors, would be rather laborious, but sufficient conditions can be given for the validity of all orders of performing the operations by specifying that the arguments of all Ψ functions involved in all orders of applying the operators satisfy condition (P). However, the existence of $E(\mid a_{\mu\nu} \mid^g \mid a_{rs} \mid^h)$ obviously does not, in general, imply that both $I_{\mu\nu}^{-2g} I_{rs}^{-2h} G_\beta$ and $I_{rs}^{-2h} I_{\mu\nu}^{-2g} G_\beta$ must exist. Therefore, the problem of finding the value of $E(\mid a_{\mu\nu} \mid^g \mid a_{rs} \mid^h)$, when it exists, is solved when one order of applying $I_{\mu\nu}^{-2g}$ and I_{rs}^{-2h} to G_β is found such that the arguments of the Ψ's in each successive operation satisfy (P).

Now let us consider a set of m distinct principal minors of $\mid a_{rs} \mid$ characterized as follows: Denote the minor $\mid a_{\mu_\sigma\nu_\sigma} \mid$ by d_σ $(\mu_\sigma, \nu_\sigma = k_\sigma + 1, k_\sigma + 2, \cdots n;$ $\sigma = 1, 2, \cdots m)$ and $0 \leq k_1 < k_2 < \cdots < k_m < t + 1$, and let the moment-generating operator for d_σ be I_σ. Let $D_{\sigma\beta}$ be the minor $\mid A_{\gamma_\sigma\delta_\sigma\beta} \mid$, $(\gamma_\sigma, \delta_\sigma = 1, 2, \cdots k_\sigma)$. Then we have at once the following formula

$$E\left(g_\beta \prod_{\sigma=1}^{m} d_\sigma^h\right) = \bar{K} [I_m^{-2h_m} I_{m-1}^{-2h_{m-1}} \cdots I_1^{-2h_1} G_\beta] = \tag{2.59}$$

$$\bar{K}\prod_{\sigma=1}^{m}[A_{\beta}^{-h_\sigma}D_{\sigma\beta}{}^{h_\sigma}a^{h_\sigma}]\prod_{\sigma=1}^{m}\prod_{i=1}^{t-k_\sigma}[\Psi(N-n+t+1-i+2+H_{\sigma-1},2h_\sigma)]\,G_\beta$$

where $H_\sigma=\sum_{\alpha=1}^{\sigma}h_\alpha$ and $H_0=0$. The validity of (2.59) depends on the condition that the arguments of the Ψ's satisfy (P). The value of the joint moment $E\left(\prod_{\sigma=1}^{m}d_\sigma^{h_\sigma}\right)$ is found by setting the β's $=0$.

Another interesting class of simultaneous moments is that associated with $|a_{rs}|$ and a set of mutually exclusive principal minors, the sum of the orders being n. The general formula can be obtained in the following manner: Let the principal minor $|a_{\mu_\rho\nu_\rho}|$ be denoted by c_ρ $(\mu_\rho,\ \nu_\rho=K_{\rho-1}+1,\ K_{\rho-1}+2,\ \cdots K_{\rho-1}+k_\rho;\ K_{\rho-1}=\sum_{\alpha=1}^{\rho-1}k_\alpha,\ K_l=n;\ k_l>n-t;\ \rho=1,2,\cdots l)$ and let the moment-generating operator for c_ρ be I_ρ. Let $C_{\rho\beta}$ be the principal minor $|A_{\mu_\rho\nu_\rho\beta}|$ $(\rho=1,2,\cdots l)$ and C_β the minor $|A_{\mu\nu\beta}|$ $(\mu,\ \nu=k_{l-1}+1,k_{l-1}+2,\cdots t)$. We shall assume that all of the A_{rs} except those contained in the set $\{C_{\rho\beta}\}$ are zero. Then by straightforward application of the I's (setting all β's which are not contained in the set $\{C_{\rho\beta}\}$ equal to zero immediately after applying the operator I_{rs}^{-2h}) we find,

$$(2.60)\quad E\left(g'_\beta\mid a_{rs}\mid^h\prod_{\rho=1}^{l}C_\rho^{h_\rho}\right)=\bar{K}\,[I_l^{-2h_l}I_{l-1}^{-2l-1}\cdots I_1^{-2h_1}I_{rs}^{-2h}G_\beta]=$$

$$\bar{K}C_\beta^{-h-h_l}a^{h+h_l}\prod_{\rho=1}^{l-1}[C_{\rho\beta}^{-h-h_\rho}]\,G_\beta\prod_{i=1}^{t}[\Psi(N-n+t+1-i,2h)]$$

$$\times\prod_{i=1}^{k_l-n+t}[\Psi(N-n+t+1+2h-i,2h_l)]$$

$$\times\prod_{\rho=1}^{l-1}\prod_{i=1}^{k_\rho}[\Psi(N+1+2h-i,2p_\rho)],$$

where g'_β is the value of g_β with all β's equal to zero except those contained in $\{C_{\rho\beta}\}$. Again we must impose the condition (P) on the arguments of all Ψ's. The value of $E\left(\mid a_{rs}\mid^h\prod_{\rho=1}^{l}C_\rho^{h_\rho}\right)$ is found from (2.60) by setting the remaining β' s $=0$.

It will be noticed that the restriction $k_l>n-t$ is not a necessary one, but was made for convenience, because the result that would be obtained by removing the restriction would be only trivially different from (2.60).

The derivation of an expression for the value of the left side of (2.60) when the

When g is a multiple of $\frac{1}{2}$ the application of $I_{\mu\nu}^{2g}$ to $(I_{rs}^{-2h}G_\beta)$ is simply to perform $2g$ times the operation of introducing into $(I_{rs}^{-2h}G_\beta)$ a 1-fold set of ξ-parameters associated with $\| a_{\mu\nu} \|$, integrating with respect to the ξ's and multiplying by $\pi^{-\frac{n-k}{2}}$. This compound operation will be denoted by

$$I_{\mu\nu}^{2g} I_{rs}^{-2h} G_\beta. \tag{2.57}$$

It can be shown, as in the case of a single operator, that the application of $I_{\mu\nu}^{-2g}$ to (2.56) yields

$$E(g_\beta \mid a_{rs} \mid^h \mid a_{\mu\nu} \mid^g) = \bar{K} \prod_{i=1}^{t} [\Psi(N - n + t + 1 - i, 2h)] \tag{2.58}$$
$$\times \prod_{i=1}^{t-k} [\Psi(N - n + t + 1 - i + 2h, 2g)] \times A_\beta^{-h-g} \mid A_{bc\beta} \mid^g a^{h+g} G_\beta,$$

and the result holds for all values of g and h for which condition (P) is satisfied in all of the Ψ's, where (P) is the condition that $a + b > 0$ and $a > 0$ in $\Psi(a, b)$. To find the simultaneous moment $E(\mid a_{rs} \mid^h \mid a_{\mu\nu} \mid^g)$ we set the β's $= 0$.

There is no difficulty involved in the above process if the operators are applied in the order just given. However, if the reverse order or any mixed order (e.g. $I_{rs}^{-2h_1} I_{\mu\nu}^{-2g_1} I_{rs}^{-2h_2} I_{\mu\nu}^{-2g_2} G_\beta$ where $h_1 + h_2 = h$ and $g_1 + g_2 = g$) is used, several troublesome features may be encountered. Firstly, $I_{\mu\nu}^{-2g} G_\beta$ will not exist if $N - n + k + 1 + 2g \leq 0$, whereas, under such conditions (2.54) will exist for certain values of h, and hence the further operation I_{rs}^{-2h} will have no meaning. Secondly, if $I_{\mu\nu}^{-2g} G_\beta$ does exist, then unless h is a multiple of $-\frac{1}{2}$ the meaning and validity of the operation I_{rs}^{-2h} $(I_{\mu\nu}^{-2g} G_\beta)$ would have to be treated by a procedure similar to that used in considering $I_{\mu\nu}^{\lambda} G_\beta$ when λ is a negative integer. A detailed treatment of the question of commutability of these operations, especially in the general case of m principal minors, would be rather laborious, but sufficient conditions can be given for the validity of all orders of performing the operations by specifying that the arguments of all Ψ functions involved in all orders of applying the operators satisfy condition (P). However, the existence of $E(\mid a_{\mu\nu} \mid^g \mid a_{rs} \mid^h)$ obviously does not, in general, imply that both $I_{\mu\nu}^{-2g} I_{rs}^{-2h} G_\beta$ and $I_{rs}^{-2h} I_{\mu\nu}^{-2g} G_\beta$ must exist. Therefore, the problem of finding the value of $E(\mid a_{\mu\nu} \mid^g \mid a_{rs} \mid^h)$, when it exists, is solved when one order of applying $I_{\mu\nu}^{-2g}$ and I_{rs}^{-2h} to G_β is found such that the arguments of the Ψ's in each successive operation satisfy (P).

Now let us consider a set of m distinct principal minors of $\mid a_{rs} \mid$ characterized as follows: Denote the minor $\mid a_{\mu_\sigma\nu_\sigma} \mid$ by d_σ $(\mu_\sigma, \nu_\sigma = k_\sigma + 1, k_\sigma + 2, \cdots n;$ $\sigma = 1, 2, \cdots m)$ and $0 \leq k_1 < k_2 < \cdots < k_m < t + 1$, and let the moment-generating operator for d_σ be I_σ. Let $D_{\sigma\beta}$ be the minor $\mid A_{\gamma_\sigma\delta_\sigma\beta} \mid$, $(\gamma_\sigma, \delta_\sigma = 1, 2, \cdots k_\sigma)$. Then we have at once the following formula

$$E\left(g_\beta \prod_{\sigma=1}^{m} d_\sigma^{h_\sigma}\right) = \bar{K} [I_m^{-2h_m} I_{m-1}^{-2h_{m-1}} \cdots I_1^{-2h_1} G_\beta] = \tag{2.59}$$

$$\bar{K}\prod_{\sigma=1}^{m}[A_{\beta}^{-h_\sigma}D_{\sigma\beta}{}^{h_\sigma}a^{h_\sigma}]\prod_{\sigma=1}^{m}\prod_{i=1}^{t-k_\sigma}[\Psi(N-n+t+1-i+2+H_{\sigma-1},2h_\sigma)]\,G_\beta$$

where $H_\sigma=\sum_{\alpha=1}^{\sigma}h_\alpha$ and $H_0=0$. The validity of (2.59) depends on the condition that the arguments of the Ψ's satisfy (P). The value of the joint moment $E\left(\prod_{\sigma=1}^{m}d_\sigma^{h_\sigma}\right)$ is found by setting the β's $=0$.

Another interesting class of simultaneous moments is that associated with $|a_{rs}|$ and a set of mutually exclusive principal minors, the sum of the orders being n. The general formula can be obtained in the following manner: Let the principal minor $|a_{\mu_\rho\nu_\rho}|$ be denoted by c_ρ $(\mu_\rho,\ \nu_\rho=K_{\rho-1}+1,\ K_{\rho-1}+2,\ \cdots K_{\rho-1}+k_\rho;\ K_{\rho-1}=\sum_{\alpha=1}^{\rho-1}k_\alpha,\ K_l=n;\ k_l>n-t;\ \rho=1,2,\cdots l)$ and let the moment-generating operator for c_ρ be I_ρ. Let $C_{\rho\beta}$ be the principal minor $|A_{\mu_\rho\nu_\rho\beta}|$ $(\rho=1,2,\cdots l)$ and C_β the minor $|A_{\mu\nu\beta}|$ $(\mu,\ \nu=k_{l-1}+1,k_{l-1}+2,\cdots t)$. We shall assume that all of the A_{rs} except those contained in the set $\{C_{\rho\beta}\}$ are zero. Then by straightforward application of the I's (setting all β's which are not contained in the set $\{C_{\rho\beta}\}$ equal to zero immediately after applying the operator I_{rs}^{-2h}) we find,

$$(2.60)\quad E\left(g'_\beta\mid a_{rs}\mid^h\prod_{\rho=1}^{l}C_\rho^{h_\rho}\right)=\bar{K}\,[I_l^{-2h_l}I_{l-1}^{-2l-1}\cdots I_1^{-2h_1}I_{rs}^{-2h}G_\beta]=$$

$$\bar{K}C_\beta{}^{-h-h_l}a^{h+h_l}\prod_{\rho=1}^{l-1}[C_{\rho\beta}^{-h-h_\rho}]\,G_\beta\prod_{i=1}^{t}[\Psi(N-n+t+1-i,2h)]$$

$$\times\prod_{i=1}^{k_l-n+t}[\Psi(N-n+t+1+2h-i,2h_l)]$$

$$\times\prod_{\rho=1}^{l-1}\prod_{i=1}^{k_\rho}[\Psi(N+1+2h-i,2p_\rho)],$$

where g'_β is the value of g_β with all β's equal to zero except those contained in $\{C_{\rho\beta}\}$. Again we must impose the condition (P) on the arguments of all Ψ's. The value of $E\left(\mid a_{rs}\mid^h\prod_{\rho=1}^{l}C_\rho^{h_\rho}\right)$ is found from (2.60) by setting the remaining β' s $=0$.

It will be noticed that the restriction $k_l>n-t$ is not a necessary one, but was made for convenience, because the result that would be obtained by removing the restriction would be only trivially different from (2.60).

The derivation of an expression for the value of the left side of (2.60) when the

A_{rs} not contained in $\{C_{\rho\beta}\}$ are not assumed to be zero is extremely complicated, and only one very special case has been worked out which will be given in the section on Applications in connection with the treatment of the multiple correlation coefficient.

III. The combination of sets of component product moments

In the previous sections we have dealt with methods of finding the moments of certain functions of the determinant and its principal minors of the total product moments $\{a_{rs}\}$ and were finally lead to the two types of moment formulas (2.59) and (2.60). In the present section we shall analyze $\| a_{rs} \|$ into a sum of matrices such that the elements of each matrix are distributed independently of those of the other matrices, and consider the problem of finding the sampling moments of various functions of the determinants of these matrices and their sums.

Let it be assumed that a_{rs} $(r, s = 1, 2, \cdots n)$ can be written as $\sum_{\gamma=1}^{\tau} a_{rs\gamma}$ where $a_{rs\gamma}$ is a bilinear form in $x_{r\alpha}$ and $x_{s\alpha}$ $(\alpha = 1, 2, \cdots N)$ and such that

$$(3.1) \qquad \int e^{-\Sigma A_{rs}a_{rs} - \Sigma\beta_{rs\gamma}a_{rs\gamma}} d\bar{x}_{i\alpha} = \pi^{\frac{Nt}{2}} \prod_{\gamma=1}^{\tau} \left[A_{\beta\gamma}^{-\frac{N\gamma}{2}} e^{-\Sigma B_{pq\beta\gamma}a_{pq\gamma}} \right]$$

where $\sum_{\gamma=1}^{\tau} N_\gamma = N$, $\beta_{rs\gamma} \equiv \beta_{sr\gamma}$, $A_{\beta\gamma} = | A_{ij} + \beta_{ij\gamma} |$, the β's are such that $|A_{ij} + \beta_{ij\gamma}|$ $(\gamma = 1, 2, \cdots \tau)$ is positive definite, $B_{pq\beta\gamma}$ is the value of B_{pq} after the substitution $A_{rs} : \rightarrow A_{rs} + \beta_{rs\gamma} = A_{rs\beta\gamma}$ $(r, s = 1, 2, \cdots n)$ has been applied. The existence of such sets as $\{a_{rs\gamma}\}$ can be verified at once from the fact that the sample of values $\{x_{r\alpha}\}$ $(r = 1, 2, \cdots n;\ \alpha = 1, 2, \cdots N)$, being drawn from an infinite population, may be treated as τ independent samples if $N > \tau$ so that the variates in the γ-th sample would be that part of the set $\{x_{r\alpha}\}$ for which α takes on N_γ values (the values being different for each value of γ), and $r_{rs\gamma}$ would be $\sum_{\alpha} x_{r\alpha}x_{s\alpha}$ where α ranges over the same N_γ values. Another well known example is the case of $\tau = 2$ in which $a_{rs1} = \Sigma_\alpha(x_{r\alpha} - \bar{X}_r)(x_{s\alpha} - \bar{X}_s)$ and $a_{rs2} = N\bar{X}_r\bar{X}_s$, where $\bar{X}_\delta = 1/N \sum_{\alpha} x_{\delta\alpha}$ $(\delta = r, s)$, $N_1 = N - 1$, $N_2 = 1$. The τ sets $\{a_{rs\gamma}\}$ $(\gamma = 1, 2, \cdots \tau)$, are said to be mutually independent when they satisfy the relation (3.1), and the joint probability function of the τ sets will be the product of τ probability functions, the γ-th one being that associated with $\{a_{rs\gamma}\}$. The sets $\{a_{rs\gamma}\}$ will be referred to as *component* sets of product moments.

Now let us write

$$G_{\beta\gamma} = \pi^{\frac{N_\gamma t}{2}} A_{\beta\gamma}^{-\frac{N_\gamma}{2}} e^{-\sum_{p,q} B_{pq\beta\gamma}a_{pq\gamma}}$$

and

$$P_{\beta\gamma}(\bar{a}_{rs\gamma}) = e^{-\sum_{r,s} A_{rs\beta\gamma}\bar{a}_{rs\gamma}}.$$

Then (3.1) can be written as

$$\int \prod_{\gamma=1}^{\tau} P_{\beta\gamma}(\bar{a}_{rs\gamma})\, d\bar{x}_{i\alpha} = \prod_{\gamma=1}^{\tau} G_{\beta\gamma}. \tag{3.2}$$

Therefore, any variation in the $\beta_{rs\gamma}$ in $P_{\beta\gamma}$ will occur on the right only in $G_{\beta\gamma}$. Hence, if we replace A_{rs} in (2.13) by $\beta_{rs\gamma}$ and apply the resulting substitution to (3.2) we get an expression in which the ξ-parameters are involved only in $P_{\beta\gamma}$ and $G_{\beta\gamma}$ and which is valid for all values of the ξ's. Similarly, we can introduce ξ-parameters into all of the P's and the corresponding G's. The fact that we can introduce ξ's into any of the pairs $(P_{\beta\gamma}, G_{\beta\gamma})$ $(\gamma = 1, 2, \cdots \tau)$ independently of all of the others is all that we require (besides conditions (a) and (b) of Theorem I) to be able to apply the I operators to any $P_{\beta\gamma}$ and the corresponding $G_{\beta\gamma}$, just as they were applied to (2.34). In fact, each of the pairs $(P_{\beta\gamma}, G_{\beta\gamma})$ has essentially the same structure as (P_β, G_β) in (2.34), that is, the set $\{a_{rs\gamma}\}$, as far as its sampling properties is concerned, is equivalent to a set of total product moments in a sample of N_γ individuals.

We can make any k-fold set of ξ's occur in any number of the pairs $(P_{\beta\gamma}, P_{\beta\gamma})$ we please by using the proper substitutions for the $\beta_{rs\gamma}$, which makes it possible to use the moment-generating operators on various combinations of P's and their corresponding G's. For convenience let $N_1 \geq N_2 \geq \cdots \geq N_p$, and let the moment-generating operator for $\| a_{rs\gamma} \|$ be $I_{(\gamma)}$ $(\gamma = 1, 2, \cdots \tau)$. Then we can write (for $s \leq \tau$)

$$E\left[\prod_{\gamma=1}^{\tau} (g_{\beta\gamma} \mid a_{rs\gamma} \mid^{h\gamma})\right] = \tag{3.3}$$

$$\bar{K}\, (I_{(1)}^{-2h_1} G_{\beta_1})\, (I_{(2)}^{-2h_2} G_{\beta_2}) \cdots (I_{(s)}^{-2h_s} G_{\beta_s}) \cdots G_{\beta_\tau}$$

where $g_{\beta\gamma} = e^{-\sum_{r,s} a_{rs\gamma}\beta_{rs\gamma}}$. But

$$I_{(\nu)}^{-2h\gamma} G_{\beta\gamma} = \prod_{i=1}^{t} |\Psi(N_\gamma - n + t + 1 - i, 2h_\gamma)|\, A_{\beta\gamma}^{-h\gamma} a_\gamma^{h\gamma} G_{\beta\gamma} \tag{3.4}$$

where $a_\gamma = | a_{pq\gamma} |$. Furthermore,

$$I_{(\gamma)}^{-2h\gamma} P_{\beta\gamma}(\bar{a}_{rs\gamma}) = e^{-\sum_{r,s} A_{rs\beta\gamma}\bar{a}_{rs\gamma}} \mid a_{rs} \mid^{h\gamma}. \tag{3.5}$$

Again, the validity of (3.3) depends on the condition that (P) be satisfied by the arguments of the Ψ's, and that $\| a_{rs\gamma} \|$ $(\gamma = 1, 2, \cdots \tau)$ be positive definite.

We can now apply slightly more general operators to (3.3) which will generate the moments of the determinants of the sum of any number of the matrices

$|| a_{rs\gamma} ||$ if the moments exist. For example, suppose we wish to find the value of the h-th moment of $| a_{rs1} + a_{rs2} |$ simultaneously with all of the moments found in (3.3); that is, to find the value of

$$E\left[\prod_{\gamma=1}^{\tau} (| a_{rs\gamma} |^{h\gamma}) \, | a_{rs1} + a_{rs2} |^{h}\right]. \tag{3.6}$$

To do this let $I_{(12)}$ be an operator which will make

$$|| \beta_{rs1} || \equiv || \beta_{rs2} || = || \beta_{rs} || \tag{3.7}$$

afterwards assuming the rôle of the ordinary I operator on the β_{rs}. Then we have,

$$I_{(12)}^{-2h} [(I_{(1)}^{-2h_1} P_{\beta_1}) (I_{(2)}^{-2h_2} P_{\beta_2})] = e^{-\Sigma A_{rs}\beta(a_{rs1}+a_{rs2})} \, | a_{rs1} |^{h_1} \, | a_{rs2} |^{h_2} \, | a_{rs1} + a_{rs2} |^{h}. \tag{3.8}$$

Therefore the value of (3.6) is given by

$$\bar{K} \; \{I_{(12)}^{-2h} [(I_{(1)}^{-2h_1} G_{\beta_1}) (I_{(2)}^{-2h_2} G_{\beta_2})]\} (I_{(3)}^{-2h_3} G_{\beta_3}) \cdots (I_{(s)}^{-2h_s} G_{\beta_s}) G_{\beta_{s+1}} \cdots G_{\beta_\tau} \tag{3.9}$$

where

$$I_{(12)}^{-2h} [(I_{(1)}^{-2h_1} G_{\beta_1}) (I_{(2)}^{-2h_2} G_{\beta_2})] = \tag{3.10}$$

$$\prod_{\gamma=1}^{2} \prod_{i=1}^{t} [\Psi(N_\gamma - n + t + 1 - i, 2h_\gamma)]$$

$$\times \prod_{i=1}^{t} [\Psi(N_1 + N_2 - n + t + 1 + 2h_1 + 2h_2 - i, 2h)]$$

$$\times G_{\beta_{(1,2)}} A_{\beta_{(1,2)}}^{-h_1-h_2-h} a_1^{h_1} a_2^{h_2} \, | a_{pq1} + a_{pq2} |^{h}$$

in which $G_{\beta_{(1,2)}} = G_{\beta_1} G_{\beta_2}$ with the condition that (3.7) is satisfied $A_{\beta_{(12)}} = | A_{ij} + \beta_{ij} |$ and $a_\gamma = | a_{pq\gamma} |$ $(\gamma = 1, 2)$.

In general, if $\sum_{\gamma=1}^{\tau} a_{rs\gamma}$ be represented as k mutually exclusive sums

$$\Sigma_{\gamma 1} a_{rs\gamma 1} + \Sigma_{\gamma 2} a_{rs\gamma 2} + \cdots + \Sigma_{\gamma k} a_{rs\gamma k} \tag{3.11}$$

$(r, s = 1, 2, \cdots n)$, then by straightforward application of the $I_{(12)}$ type of operator on (3.3) we can find the value of

$$E\left[\prod_{\gamma=1}^{\tau} (| a_{rs\gamma} |^{h\gamma}) \prod_{i=1}^{k} | \Sigma_{\gamma i} a_{rs\gamma i} |^{gi}\right]. \tag{3.12}$$

In fact, we can regroup the k sums in (3.11) into mutually exclusive groups and so on and get further generalizations. We shall not write out the value of (3.12) but only consider special cases of it in the section on Applications when dealing

with the generalized "Student" ratio and the Neyman-Pearson λ_H criterion for k samples.

IV. Applications of the operators to sampling problems

1. **Generalized variance of deviations from regression functions.** We shall suppose that the variate x_i $(i = 1, 2, \cdots t)$ is normally distributed about the regression function $\sum_p C_{ip}x_p$ such that the joint probability law of $\{x_i\}$ is

$$\frac{A^{\frac{1}{2}}}{\pi^{\frac{t}{2}}} e^{-\sum A_{ij}(x_i - \sum_p C_{ip}x_p)(x_j - \sum_p C_{jp}x_p)} \tag{4.1}$$

where $A = |A_{ij}|$, $||A_{ij}||$ is positive definite, the C_{ip} are any constants and the x's are deviations from means, that is $x_r = X_r - m_r$ $(r = 1, 2, \cdots n)$. The x_p may be functions of one or more variables; for example, in the analysis of time series they may be, for computational convenience, orthogonal functions of t over a set of values, the most important case being that in which the functions are polynomials[7]. If we let $||\bar{A}^{ij}||$ be the inverse of the matrix $||2A_{ij}||$ then $\bar{A}^{ij} = \rho_{ij}\sigma_i\sigma_j$ where σ_i is the population value of the standard deviation of the deviations of x_i from its regression function, and ρ_{ij} is the population value of the correlation coefficient between the deviations of x_i and x_j from their regression functions. Hence, A_{ij}, when expressed in terms of the inverse notation, is $\frac{1}{2}\frac{\Delta_{ij}}{\Delta\sigma_i\sigma_j}$ where $\Delta = |\rho_{ij}|$ and Δ_{ij} is the cofactor of ρ_{ij} in Δ.

Now let a sample Σ of N individuals be drawn from the universe specified by (4.1), and as before, denote the value of x_i obtained in the α-th drawing by $x_{i\alpha}$ when the value assigned to x_p is $x_{p\alpha}$. The probability law of the sample Σ is

$$\pi^{-\frac{Nt}{2}} A^{\frac{N}{2}} e^{-\theta} \tag{4.2}$$

where

$$\theta = \sum_{i,j,\alpha} A_{ij}\left(x_{i\alpha} - \sum_p C_{ip}x_{p\alpha}\right)\left(x_{j\alpha} - \sum_p C_{jp}x_{p\alpha}\right).$$

It is interesting to note that the maximum of (4.2) for variations of $m_i - \sum_p C_{ip}m_p = \bar{m}_i$ say, B_{ip} and A_{ij} $(i, j = 1, 2, \cdots t; p = t+1, t+2 \cdots n)$ occurs when

(4.2a) $\bar{m}_i = \bar{X}_i - \sum_p C'_{ip}\bar{X}_p$, $C_{ip} = \sum_q v_{iq}v^{pq} = C'_{ip}$ say, and $A_{ij} = \frac{1}{2}d^{ij}$,

[7] For a detailed treatment on this subject see, C. Jordan, Approximation and graduation according to the principle of least squares by orthogonal functions, Annals of Mathematical Statistics, Vol. III (1932) No. 4.

where $v_{rs} = \sum_{\alpha} (X_{r\alpha} - \bar{X}_r)(X_{s\alpha} - \bar{X}_s)$, $\bar{X}_\delta = 1/N \sum_{\alpha} X_{\delta\alpha}$ $(r, s = 1, 2, \cdots n;$ $\delta = r, s)$ v^{pq} is the element in the p-th row and q-th column of the inverse of $|| v_{pq} ||$, while d^{ij} is the element in the i-th row and j-th column of the inverse of $|| d_{ij} ||$ where

$$d_{ij} = 1/N \left(v_{ij} - \sum_{p,q} v_{jp} v_{iq} v^{pq} \right). \tag{4.2b}$$

The value of the maximum of (4.2) is $K_o \mid d_{ij} \mid^{-\frac{N}{2}}$ where K_o is independent of the observations $\{X_{r\alpha}\}$. The quantity[8] $\mid d_{ij} \mid$ will be called the *generalized variance* in Σ of the variates x_i about their least-square regression functions.

If we denote by $\mid v_{rs} \mid_{ij}$ the determinant obtained from $\mid v_{rs} \mid$ by striking out all of the first rows and columns except the i-th and j-th column, then $d_{ij} = 1/N \dfrac{\mid v_{rs} \mid_{ij}}{\mid v_{pq} \mid}$. Now it can be shown by the Jacobi Theorem that

$$\mid d_{ij} \mid = (1/N)^t \frac{\mid v_{rs} \mid}{\mid v_{pq} \mid}. \tag{4.3}$$

The problem is to find the value of $E(\mid d_{ij} \mid^k)$. Since N and $\mid v_{pq} \mid$ are constants, we have

$$E(\mid d_{ij} \mid^k) = N^{-tk} \mid v_{pq} \mid^{-k} E(\mid v_{rs} \mid^k) \tag{4.4}$$

so that we only need to find the moments of $\mid v_{rs} \mid$.

In order to obtain the usual form on which to apply the operators we shall find the value of $E(g_\beta)$ where $g_\beta = e^{-\Sigma \beta_{rs} v_{rs}}$ $(\beta_{rs} \equiv \beta_{sr})$, which is the integral of (4.2) when multiplied by g_β. From its definition v_{rs} can be written as

$$(1 - 1/N) \Sigma x_{r\alpha} x_{s\alpha} - 1/N \sum_{\alpha \neq \alpha'} x_{r\alpha} x_{s\alpha'}.$$

Therefore, the exponent in the integral defining $E(g_\beta)$ is a quadratic form in the Nn variates $\{x_{r\alpha}\}$ such that the part which is homogeneous in the $x_{i\alpha}$ is positive definite for sufficiently small values of the β_{ij}. If we write $\sum_i A_{ij} C_{ip} = -A_{jp}$ and $\sum_{i,j} A_{ij} C_{ip} C_{jq} = A_{pq}$ $(p, q = t + 1, t + 2, \cdots n;\ j = 1, 2, \cdots t)$ then the matrix of the quadratic in the $x_{r\alpha}$ can be expressed as a matrix $|| D_{rs} ||$ of square blocks of elements where D_{rs} is the block containing N^2 elements in which each

[8] See R. Frisch, Correlation and scatter in statistical variables, Nordisk Statistisk Tidskrift, Vol. 8 (1929), p. 52. Here Frisch gives the geometrical interpretation and significance in regression problems of a quantity which he calls the *collective standard deviation* and which is essentially the same in structure as the square root of $\mid d_{ij} \mid$ but does not consider its sampling theory.

diagonal element is $A_{rs} + \beta_{rs}(1 - 1/N)$ and all others have the value $-\beta_{rs}/N$ Hence, it follows from (1.1) that

$$E(g_\beta) = A^{\frac{N}{2}} \mid D_{ij} \mid^{-1} e^{-\tau} \tag{4.5}$$

where

$$\tau = \Sigma A_{pq\beta} v_{pq} + \Sigma A_{pq} x_{p\alpha} x_{q\alpha} + \mid D_{ij} \mid_B / \mid D_{ij} \mid,$$

$\mid D_{ij} \mid$ is the minor of $\mid D_{rs} \mid$ with $i, j = 1, 2, \cdots t$, and $\mid D_{ij} \mid_B$ is a symmetrically bordered form of $\mid D_{ij} \mid$ in which the element bordering at the right of the α-th row within the i-th row of blocks is

$$\sum_p (A_{ip} + \beta_{ip}) x_{p\alpha} - 1/N \sum_{p,\alpha} \beta_{ip} x_{p\alpha}. \tag{4.6}$$

The problem of finding $E(g_\beta)$ is therefore equivalent to that of evaluating the determinants $\mid D_{ij} \mid$ and $\mid D_{ij} \mid_B$. To evaluate $\mid D_{ij} \mid$, we subtract the second row from the first row in the i-th row of blocks, the third from the second, and so on; and repeat the process for all rows of blocks, that is, for $i = 1, 2, \cdots t$; then add the first column to the second column in the j-th column of blocks, add the second to the third and so on, repeating the process for all t columns of blocks. The resulting determinant will consist of square blocks of elements, each block having N^2 elements and such that all of the elements in each block are zero except those in the main diagonal and last row. Consequently Laplace's expansion may be used and we find at once

$$\mid D_{ij} \mid = A_\beta^{N-1} A, \tag{4.7}$$

where $A_\beta = \mid A_{ij\beta} \mid$, and $A_{ij\beta} = A_{ij} + \beta_{ij}$. In a similar manner, we find

$$\mid D_{ij} \mid_B = - A_\beta^{N-1} A \{\Sigma A_{ip\beta} A_{jq\beta} A_\beta^{ij} v_{pq} + \Sigma A_{ip} A_{jq} A^{ij} (\bar{X}_p - m_p)(\bar{X}_q - m_q)\}$$

in which A_β^{ij} is the general element in the inverse of $\| A_{ij\beta} \|$. Therefore we have for (4.5)

$$E(g_\beta) = A^{\frac{N-1}{2}} A_\beta^{-\frac{N-1}{2}} e^{-\Sigma B_{pq\beta} v_{pq}} = A^{\frac{N-1}{2}} G_\beta, \tag{4.8}$$

where $B_{pq\beta} = A_{pq\beta} - \sum_{i,j} A_{ip\beta} A_{jq\beta} A_\beta^{ij}$. It will be seen from the definitions of A_{ip} and A_{pq} that $B_{pq\beta}$ vanishes when the β's $= 0$.

If we denote the moment-generating operator associated with $\| v_{rs} \|$ by I_{rs}, then if $\| v_{pq} \|$ is positive definite we have

$$E(g_\beta \mid v_{rs} \mid^k) = A^{\frac{N-1}{2}} I_{rs}^{-2k} G_\beta = A^{\frac{N-1}{2}} \left\{ A_\beta^{-k} v^k \prod_{i=1}^{t} [\Psi(N - n + t - i, 2k)] G_\beta \right\}, \tag{4.9}$$

where $v = |v_{pq}|$. To get the k-th moment of $|v_{rs}|$ we set the β's $= 0$. Inserting the result in (4.4) we get

$$E(|d_{ij}|^k) = N^{-tk} A^{-k} \prod_{i=1}^{t} [\Psi(N - n + t - i, 2k)]. \tag{4.10}$$

We have seen that the generalized variance in the sample Σ of the x_i about their least square regression functions of the independent variates $\{x_p\}$ when expressed in terms of the v_{rs} is given by (4.3). By the same procedure it follows that if $g < t$, then the generalized variance $|d_{\gamma\delta}|$ say, in Σ, of $x_1, x_2, \cdots x_g$ about their least square regression functions of the remaining variates of the set $\{x_r\}$ is given by

$$|d_{\gamma\delta}| = N^{-g} |v_{rs}| / |v_{\mu\nu}|, \tag{4.11}$$

where $\mu, \nu = g + 1, g + 2, \cdots n$. Therefore

$$E(g_\beta | d_{\gamma\delta}|^k) = N^{-gk} E(g_\beta | v_{rs}|^k | v_{\mu\nu}|^{-k}) = N^{-gk} A^{\frac{N-1}{2}} I_{\mu\nu}^{2k} I_{rs}^{-2k} G_\beta, \tag{4.12}$$

from which we find

$$E(|d_{\gamma\delta}|^k) = N^{-gk} |A_{\gamma\delta}|^{-k} \prod_{i=1}^{g} [\Psi(N - n + g - i, 2k)], \tag{4.13}$$

where $\gamma, \delta = 1, 2, \cdots g$.

An interesting special case is that in which $\gamma, \delta = 1$, when (4.13) yields the moments of the sample variance of the deviations of the variate x_1 about its least square regression line computed from the remaining $n - 1$ variates. In fact, the k-th moment in this case is

$$N^{-k} A_{11}^{-k} \frac{\Gamma\left(\frac{N-n}{2} + k\right)}{\Gamma\left(\frac{N-n}{2}\right)}. \tag{4.14}$$

The moments given by (4.10) and (4.13) are the same in structure. Therefore, as far as the moments and sampling distribution of the generalized variance are concerned, it is immaterial whether all of the variates in the regression functions are subject to normal sampling variation or remain fixed, for the results are the same.

2. A generalized "Student" ratio for regression coefficients. If we determine the A_{ij} for given values of the $\bar{m}_i$ and C_{ip} such that (4.2) is a maximum, it is found that the maximum occurs for

$$A_{ij} = \tfrac{1}{2} c^{ij} \qquad (i, j = 1, 2, \cdots t), \tag{4.15}$$

where

$$c_{ij} = 1/N \sum_{\alpha} \left(x_{i\alpha} - \sum_{\rho} C_{ip}\, x_{p\alpha} \right) \left(x_{j\alpha} - \sum_{\rho} C_{jp}\, x_{p\alpha} \right).$$

Now the ratio of the maximum of (4.2) under the conditions just given to the maximum of (4.2) for variations of the A's, $\bar{m}$'s and C's is a constant multiple of the $N/2$-th power of $| d_{ij} | / | c_{ij} | = \zeta$ say. ζ is a criterion of the Neyman-Pearson[9] λ type which may be used for testing the hypothesis H that the sample Σ has come from a normal population having specified values of the $\bar{m}_i$ and C_{ip} when it is known to have come from a normal population having a distribution law of the functional form (4.1).

To find the moments of ζ we set $Nc_{ij} = \bar{c}_{ij}$ and

$$g_{\beta,\gamma} = e^{-\Sigma \beta_{rs} v_{rs} - \Sigma \gamma_{ij} \bar{c}_{ij}} \qquad (\gamma_{ij} \equiv \gamma_{ji})$$

and determine the value of $E\,(g_{\beta,\gamma})$ which is found by making the substitution $A_{ij} : \rightarrow A_{ij} + \gamma_{ij}$ in (4.8) everywhere except in the factor $A^{\frac{N}{2}}$. Therefore

$$E\,(g_{\beta,\gamma}) = A^{\frac{N}{2}}\, A_{\gamma}^{-\frac{1}{2}}\, A_{\beta,\gamma}^{-\frac{N-1}{2}}\, e^{-\Sigma B_{pq} v_{pq}} = A^{\frac{N}{2}}\, G_{\beta,\gamma} \tag{4.16}$$

where $A_{\beta,\gamma} = | A_{ij} + \beta_{ij} + \gamma_{ij} |$, $A_{\gamma} = | A_{ij} + \gamma_{ij} |$ and B_{pq} is the value of $B_{pq\beta}$ after applying the substitution $A_{ij} : \rightarrow A_{ij} + \gamma_{ij}$. Denoting the moment-generating operator for $| \bar{c}_{ij} |$ by $I_{(\overline{ij})}$ we have at once

$$E\,(\zeta^h) = A^{\frac{N}{2}}\, v^{-h} \{ I_{(\overline{ij})}^{2h} [I_{rs}^{-2h}\, G_{\beta,\gamma}]_{\bar{\beta}=o} \}_{\bar{\gamma}=o}. \tag{4.17}$$

Performing the operation $I_{rs}^{-2h}\, G_{\beta,\gamma}$ and afterwards setting the β's $= 0$, we get for the expression in the square brackets

$$A^{-\frac{N}{2}-h}\, v^{-h} \prod_{i=1}^{t} [\Psi(N - n + t - i, 2h)]. \tag{4.18}$$

Operating on (4.18) with $I_{(\overline{ij})}^{2h}$ afterwards setting the γ's $= 0$ and multiplying by the constant factor $v^{-h} A^{\frac{N}{2}}$, we finally obtain

$$E\,(\zeta^h) = \prod_{i=1}^{t} [\Psi(N - n + t - i, 2h)\, \Psi(N + 1 + 2h - i, -2h)]. \tag{4.19}$$

An important special case of (4.19) arises when $t = n$, that is, when there are no independent variates, in which case ζ becomes the ratio

$$\frac{| v_{ij} |}{| \bar{c}_{ij} |} = \frac{1}{1 + \dfrac{T^2}{N - 1}} \tag{4.20}$$

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[9] See J. Neyman and E. S. Pearson, On the problem of the most efficient tests of statistical hypotheses, Philosophical Transactions of the Royal Society of London, Series A, Vol. 231 (1933) p. 295.

where $\bar{c}_{ij} = v_{ij} + N(\bar{X}_i - m_i)\ (\bar{X}_j - m_j)$ and $T^2 = N - 1/N\ \Sigma\, v^{ij}\ (\bar{X}_i - m_i) \times (\bar{X}_j - m_j)$, which is Hotelling's[10] generalized "Student" ratio. The h-th moment of this ratio is clearly

$$\Psi(N + 2h, -2h)\ \Psi\ (N - n, 2h)$$

from which the distribution of T^2 has been found by another method.[11]

3. **The multiple correlation coefficient.** The correlation coefficient between a variate X_1 and that linear function of the variates $X_2, X_3, \cdots X_n$ which maximizes the correlation is known as the multiple correlation coefficient. If R is the sample value of the multiple correlation coefficient in Σ between X_1 and the remaining variates then

$$1 - R^2 = \frac{|\ v_{rs}\ |}{v_{11}\ \bar{v}_{11}}, \tag{4.21}$$

where $\bar{v}_{11}$ is the cofactor of v_{11} in $|\ v_{rs}\ |$. Then since

$$E\ [(1 - R^2)^k] = E\ [|\ v_{rs}\ |^k\ v_{11}^{-k}\ \bar{v}_{11}^{-k}] \tag{4.22}$$

the problem of finding $E\ [(1 - R^2)^k]$ is simply that of operating on (4.8) with I_{rs}, $\bar{I}_{11}$ and I_{11} where $\bar{I}_{11}$ and I_{11} are the operators for $\bar{v}_{11}$ and v_{11} respectively. Therefore

$$E\ [(1 - R^2)^k] = [A^{\frac{N-1}{2}}\ \bar{I}_{11}^{2k}\ I_{11}^{2k}\ I_{rs}^{-2k}\ G_\beta]. \tag{4.23}$$

The value of $I_{rs}^{-2k}\ G_\beta$ is given by the expression in the brace in (4.9). Operating further with $\bar{I}_{11}^{2k}$ we find

$$E\ [(1 - R^2)^k] = [A^{\frac{N-1}{2}}\ I_{11}^{2k}\ \{A_{11\beta}^{-k}\ \Psi(N - n, 2k)\ G_\beta\}\ ]_{\bar{\beta}=0}. \tag{4.24}$$

Since I_{11} operates only through β_{11} we shall set the remaining β's $= 0$ before using I_{11}. Let $B_{pq\beta_1}$ be the value of $B_{pq\beta}$ when all β's are zero except β_{11}. Then

$$B_{pq\beta_1} = \frac{1}{A + \beta_{11}\ A_{11}}\ [A\ B_{pq} + \beta_{11}\ \bar{A}_{11}\ B'_{pq}] \tag{4.25}$$

where

$$B'_{pq} = A_{pq} - \sum_{g,h} A_{gq} A_{ph} A_1^{gh} \tag{4.26}$$

in which g and h take the values $2, 3, \cdots t$ and A_1^{gh} is the general element in the inverse of the matrix obtained by striking out the first row and column of $||\ A_{ij}\ ||$.

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[10] H. Hotelling, The generalization of Student's ratio, Annals of Mathematical Statistics, Vol. II (1931) pp. 359–378.

[11] S. S. Wilks, *loc. cit.* p. 487.

Remembering that $A_{jp} = \sum_{i,j} A_{ij}C_{ip}C_{jq}$ and inserting in (4.26) we find by straightforward reduction that

$$B'_{pq} = A/\bar{A}_{11}C_{1p}C_{1q}. \tag{4.27}$$

Therefore we have from the structure of v_{pq} and (4.27) that $\sum_{p,q} B'_{pq}v_{pq}$ is non-negative. Denoting this by $\frac{1}{2}\gamma^2$ and using the fact that the B_{pq} are zero we have

$$\Sigma B_{pq\beta 1} = \tfrac{1}{2}\gamma^2 - \frac{H\frac{\gamma^2}{2}}{H+\beta_{11}} \tag{4.28}$$

where $H = A/A_{11}$.

In order to get (4.24) into a form which can be operated on by I_{11} let us write

$$H + \beta_{11} = (A_{11} + \beta_{11})\left(1 - \frac{A_{11} - H}{A_{11} + \beta_{11}}\right). \tag{4.29}$$

Substituting (4.28) and (4.29) in (4.24) and setting all β's $= 0$ except β_{11} we have for the quantity in the brace

$$\bar{A}_{11}^{-\frac{N-1}{2}}\,(A_{11}+\beta_{11})^{-\frac{N-1+2k}{2}}\left(1-\frac{A_{11}-H}{A_{11}+\beta_{11}}\right)^{-\frac{N-1}{2}} e^{-\frac{1}{2}\gamma^2+\varphi}\,\Psi(N-n,2k) \tag{4.30}$$

where

$$\varphi = \frac{\frac{\gamma^2}{2}H}{(A_{11}+\beta_{11})\left(1-\frac{A_{11}-H}{A_{11}+\beta_{11}}\right)}.$$

Next we expand e^{φ} into the Taylor series $\Sigma\,\varphi^i/i!$ which converges uniformly in β_{11} over the interval $(0,\infty)$. Since $||\,A_{ij}\,||$ is positive definite $0 < H < A_{11}$ and hence $\dfrac{A_{11}-H}{A_{11}+\beta_{11}} < 1$ for $\beta_{11} \geq 0$. Therefore $\left(1-\dfrac{A_{11}-H}{A_{11}+\beta_{11}}\right)^{-m}$ $(m>0)$ can be expanded into the series $\dfrac{1}{\Gamma(m)}\sum_{i=0}^{\infty}\left(\dfrac{A_{11}-H}{A_{11}+\beta_{11}}\right)^i \dfrac{\Gamma(m+i)}{i!}$ which again converges uniformly in β_{11} over $(0,\infty)$. From the double expansion we get for (4.30) the series

$$\bar{A}_{11}^{-\frac{N-1}{2}} e^{-\frac{\gamma^2}{2}} \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\left(\frac{\gamma^2 H}{2}\right)^{\mu}(A_{11}-H)\,(A_{11}+\beta_{11})^{-\mu-\nu-\frac{N-1+2k}{2}}}{\mu!\,\nu!} \tag{4.31}$$

$$\times\,\Psi(N-1+2\mu,2\nu)\,\Psi(N-n,2k)$$

which converges uniformly in β_{11} on $(0,\infty)$. Therefore I_{11} can be applied term

by term to (4.31), which is a series in the reciprocal of the first order determinant $(A_{11} + \beta_{11})$. Applying I_{11}^{2k} to the general term we get

$$(4.32) \qquad I_{11}^{2k}\left[(A_{11}+\beta_{11})^{-\mu-\nu-\frac{N-1+2k}{2}}\right]_{\beta_{11}=0} = A_{11}^{-\mu-\nu-\frac{N-1}{2}} \times \Psi(N-1+2\mu+2\nu+2k, -2k),$$

and finally obtain for (4.24)

$$(4.33) \quad E[(1-R^2)^k] = (H/A_{11})^{\frac{N-1}{2}} e^{-\frac{\gamma^2}{2}} \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\left(\frac{\gamma^2 H}{2A_{11}}\right)^{\mu}\left(1-\frac{H}{A_{11}}\right)^{\nu}}{\mu!\,\nu!} \times \Psi(N-1+2\mu, 2\nu) \times \Psi(N-1+2\mu+2\nu+2k, -2\mu-2\nu-n+1) \times \Psi(N-n, n-1+2\mu+2\nu),$$

which converges for all values of k for which $N - n + 2k > 0$ and $N - 1 + 2k > 0$. The quantity H/A_{11} is $1 - \rho^2$ where ρ is the multiple correlation coefficient in the population between the deviations of x_1 from its regression function and the deviations of $x_2, x_3, \cdots x_t$ from their corresponding regression functions.

To find the sampling distribution of R^2 from (4.33) we use the relation

$$(4.34) \quad \Psi(N-1+2\mu+2\nu+2k, -2\mu-2\nu-n+1) = \frac{1}{\Gamma\left(\frac{n-1}{2}+\mu+\nu\right)} \times \int_0^1 (1-\theta)^{\frac{N-n}{2}+k-1}\,\theta^{\frac{n-3}{2}+\mu+\nu}\,d\theta$$

in (4.33). Interchanging the order of summation and integration we get

$$(4.35) \quad E[(1-R^2)^k] = (1-\rho^2)^{\frac{N-1}{2}} e^{-\frac{\gamma^2}{2}} \int_0^1 \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\left[\frac{\gamma^2}{2}(1-\rho^2)\right]^{\mu}\rho^{2\nu}(1-\theta)^{\frac{N-n}{2}+k-1}\,\theta^{\frac{n-3}{2}+\mu+\nu}\,\Gamma^2\left(\frac{N-1}{2}+\mu+\nu\right)d\theta}{\mu!\,\nu!\,\Gamma\left(\frac{N-n}{2}\right)\Gamma\left(\frac{N-1}{2}+\mu\right)\Gamma\left(\frac{n-1}{2}+\mu+\nu\right)}.$$

It is clear that the k-th moment of $(1 - R^2)$ is identical with that of $1 - \theta$, where the distribution function of θ is regarded as the product of $(1-\rho^2)^{\frac{N-1}{2}} \cdot e^{-\frac{\gamma^2}{2}}$ and the integrand of (4.35) with $k = 0$. The distribution of θ is the solution of a moment problem for a finite interval and hence[12] the distribution of R^2 is

[12] Cf. W. Stekloff, Mémoire de l'Académie Impériale des Sciences de St. Petersbourg, Vol. XXXII, No. 9, 1914.

uniquely determined and is identical with that of θ. That is,

$$F(R^2)\,dR^2 = \frac{(1-\rho^2)^{\frac{N-1}{2}} e^{-\frac{\gamma^2}{2}} (1-R^2)^{\frac{N-n}{2}-1} (R^2)^{\frac{n-3}{2}}}{\Gamma\left(\frac{N-n}{2}\right)} \tag{4.36}$$

$$\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \frac{\left[\frac{\gamma^2}{2}(1-\rho^2)\right]^{\mu} \rho^{2\nu} (R^2)^{\mu+\nu} \Gamma^2\left(\frac{N-1}{2}+\mu+\nu\right) dR^2}{\mu!\,\nu!\,\Gamma\left(\frac{N-1}{2}+\mu\right)\Gamma\left(\frac{n-1}{2}+\mu+\nu\right)}.$$

It will be noticed that this is the sampling distribution of the multiple correlation coefficient between a variate x_1 and $n-1$ variates of which $t-1$ are subject to sampling variation and the remaining ones are fixed. By widely different methods Fisher[13] has arrived at the distribution of R^2 for the two extreme cases in which all independent variates are (1) subject to sampling variation, and (2) fixed, and denotes the distributions for the two cases by (A) and (C) respectively. It can be readily verified that in case (1) $t = n$, $\gamma = 0$ in (4.36) and we get Fisher's (A) distribution and in case (2) $t = 1$, $H = A_{11}$ $\rho = 0$ and we get his (C) form. The (A) form has also been derived[14] directly from Wishart's generalized product moment distribution.

4. **The generalized Neyman-Pearson λ_H criterion for k samples.** Suppose that each of k samples Σ_γ ($\gamma = 1, 2, \cdots k$) has been drawn from a universe with a probability law of the form (4.1), and let the number of individuals in the γ-th sample be N_γ. Then the probability law of the k samples taken jointly will be

$$\prod_{\gamma=1}^{k} P_\gamma \tag{4.37}$$

where P_γ is (4.2) with N, the A_{ij}, C_{ip} and m_r specified by attaching a γ subscript.

Let $\bar{m}_{i\gamma} = m_{i\gamma} - \sum_p C_{ip\gamma}\, m_{p\gamma}$ and consider the maximum of (4.37) for independent variations of $A_{ij\gamma}$, $B_{ip\gamma}$ and $m_{i\gamma}$ ($i, j = 1, 2, \cdots t$; $p = t+1, t+2, \cdots n$; $\gamma = 1, 2, \cdots k$). Denoting the maximum of (4.37) under these conditions by $\Omega(\max)$ we find

$$\Omega(\max) = J \prod_{\gamma=1}^{k} |\, d_{ij\gamma}\,|^{-\frac{N_\gamma}{2}} \tag{4.38}$$

where J is independent of the x's, and $|\, d_{ij\gamma}\,|$ is given by (4.3) when (4.3) is specified by subscripts for the γ-th sample.

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[13] R. A. Fisher, The general sampling distribution of the multiple correlation coefficient, Proceedings of the Royal Society of London, Series A, Vol. CXXI (1928) pp. 654–673.

[14] S. S. Wilks, On the sampling distribution of the multiple correlation coefficient, Annals of Mathematical Statistics, Vol. III, (1932) pp. 196–203.

Now suppose $A_{ij\gamma} = A_{ij}, C_{ip\gamma} = C_{ip}$ and $\bar{m}_{i\gamma} = \bar{m}_i$ ($\gamma = 1, 2, \cdots k$), that is, that the k populations have the same set of parameters. The maximum of (4.37) under these conditions will be denoted by ω (max) and is

$$\omega(\max) = J \mid d_{ijo} \mid^{-\frac{N}{2}} \tag{4.39}$$

where $N = \Sigma N_\gamma$ and $\mid d_{ijo} \mid$ is the value given by (4.3) for the pool of the k samples $\{\Sigma_\gamma\}$. Now let us write

$$\lambda_H = \frac{\omega(\max)}{\Omega(\max)} = \frac{\prod_{\gamma=1}^{k} \mid d_{ij\gamma} \mid^{\frac{N\gamma}{2}}}{\mid d_{ijo} \mid^{\frac{N}{2}}}. \tag{4.40}$$

λ_H is the generalized Neyman-Pearson[15] likelihood criterion for testing the hypothesis H that all of the samples Σ_γ come from normal populations with the same system of parameters, when it is assumed that they are all from normal populations whose distribution functions are of the functional form (4.1). We note that $\lambda_H^{\frac{2}{N}}$ is the ratio of the geometric mean of the determinants of a set of positive definite matrices to the determinant of the arithmetic mean of the matrices, and has many interesting properties. But here we shall only consider the moments of λ_H when H is true.

From the expressions defining the d's we have

$$\lambda_H = \prod_{\gamma=1}^{k} \left[\left(\frac{N}{N_\gamma} \right)^{\frac{tN_\gamma}{2}} \left(\frac{\mid v_{pqo} \mid}{\mid v_{pq\gamma} \mid} \right)^{\frac{N_\gamma}{2}} \bar{\lambda}_H \right] \tag{4.41}$$

where $\bar{\lambda}_H = \prod_{\gamma=1}^{k} \left(\frac{\mid v_{rs\gamma} \mid}{\mid v_{rso} \mid} \right)^{\frac{N_\gamma}{2}}$, $v_{rs\gamma}$ ($r, s = 1, 2, \cdots n$) is defined as in (4.2a) for the γ-th sample, and v_{rso} is the same function for the sample formed by combining the samples $\{\Sigma_\gamma\}$. Now, since $\mid v_{pqo} \mid$ and the $\mid v_{pq} \mid$ involve only the independent variates, they are constants as far as the moments of λ_H are concerned. Therefore, we are to find the value of $E(\bar{\lambda}_H^h)$. Following the usual procedure let $g_{\beta\gamma,\beta_0} = e^{-\sum_{r,s,\gamma} \beta_{rs\gamma} v_{rs\gamma} - \Sigma \beta_{rso} v_{rso}}$ and consider

$$E\,(g_{\beta\gamma,\beta_0}). \tag{4.42}$$

The value of (4.42) can be found by the same method used in deriving (4.8). Accordingly, we get

$$E\,(g_{\beta\gamma,\beta_0}) = A^{\frac{N-1}{2}} A_{\beta_0}^{-\frac{k-1}{2}} e^{-\Sigma B_{pq\beta_0} v_{pqm}} \prod_{\gamma=1}^{k} G'_{\beta\gamma} = A^{\frac{N-1}{2}} \bar{G}_{\beta\gamma,\beta_0} \text{ say,} \tag{4.43}$$

[15] For the problem in one variate see J. Neyman and E. S. Pearson, On the problem of k samples, Bulletin de l'Académie Polonaise des Sciences et des Lettres, Serie A, Sciences Mathématiques, (1931) pp. 460–481.

where $G'_{\beta\gamma} = |A'_{ij\beta\gamma}|^{-\frac{N\gamma-1}{2}} e^{-\Sigma B'_{pq\beta\gamma} v_{pq\gamma}}$; $A_{\beta_0} = |A_{ij} + \beta_{ij0}|$; $A'_{ij\beta_0} = A_{ij\beta 0} + \beta_{ij\gamma}$; $B'_{pq\beta\gamma} = A'_{pq\beta\gamma} = \sum_{i,j} A'_{ip\beta\gamma} A'_{jq\beta\gamma} A'^{ij}_{\beta\gamma}$; $A'^{ij}_{\beta\gamma}$ is the general element in the inverse of $||A'_{ij\beta\gamma}||$; $B_{pq\beta_0}$ is the value of $B'_{pq\beta\gamma}$ with the $\beta'_\gamma s = 0$; $v_{pqm} = \sum_\gamma N_\gamma (\bar{X}_{p\gamma} - \bar{X}_p)(\bar{X}_{q\gamma} - \bar{X}_q)$ in which $\bar{X}_{p\gamma}$ is the mean of X_p in the γ-th sample Σ_γ and $\bar{X}_p$ is the mean of X_p in the sample composed by pooling $\Sigma_1, \Sigma_2, \cdots \Sigma_k$.

Now let $I_{(0)}$ be the operator associated with $||v_{rso}||$ and $I_{(\gamma)}$ that associated with $||v_{rs\gamma}||$ $(\gamma = 1, 2, \cdots k)$. Then we have,

$$E(\bar{\lambda}_H^h) = E\left[g_{\beta_\gamma,\beta_0} \prod_{\gamma=1}^{k} \left(\frac{|v_{rs\gamma}|}{v_{rso}}\right)^{\frac{N_\gamma h}{2}}\right]_{\bar{\beta}=0} = \tag{4.44}$$

$$\{I_{(0)}^{Nh} [I_{(k)}^{-N_k h} I_{(k-1)}^{-N_{k-1}h} \cdots I_{(1)}^{-N_1 h} \bar{G}_{\beta_\gamma,\beta_0}]_{\bar{\beta}_\gamma=0}\}_{\bar{\beta}_0=0},$$

where $\bar{\beta}$ is the set of all β's, $\bar{\beta}_\gamma$ the set $\{\beta_{rs\gamma}\}$ and $\bar{\beta}_o$ the set $\{\beta_{rs0}\}$. Each of the operations in the square brackets in (4.44) yields an expression of the form in the brace in (4.9). After applying $I_{(1)}, I_{(2)}, \cdots I_{(k)}$ and writing $\bar{\beta}_\gamma = 0$ we get

$$A^{\frac{N-1}{2}} \prod_{i=1}^{t} \prod_{\gamma=1}^{k} [\Psi(N_\gamma - n + t - i, N_\gamma h)] \tag{4.45}$$

$$\prod_{\gamma=1}^{k} \left[|v_{pq\gamma}|^{\frac{N\gamma h}{2}}\right] A_{\beta_0}^{-\frac{N-1+Nh}{2}} e^{-\Sigma B_{pq\beta_0} v_{pq_0}},$$

which is similar to (4.8) as far as the β's are concerned. Operating on (4.45) with $I_{(o)}^{Nh}$ and setting $\bar{\beta}_0 = 0$ we get the value of $E(\bar{\lambda}_H^h)$ and hence, from (4.41)

$$E(\lambda_H^h) = \prod_{\gamma=1}^{k} \left\{\left(\frac{N}{N_\gamma}\right)^{\frac{tN_\gamma h}{2}} \prod_{i=1}^{t} [\Psi(N_\gamma - n + t - i, N_\gamma h)]\right\} \tag{4.46}$$

$$\prod_{i=1}^{t} [\Psi(N - n + t + hN - i, -hN)].$$

The moments of generalizations of other Neyman-Pearson criteria, for example,[16] their λ_{H_1} and λ_{H_2} can be found without much difficulty in the foregoing manner.

V. Sampling distributions

It will be found that the moments of a large number of statistical functions,

[16] Cf. J. Neyman and E. S. Pearson, *loc. cit.* p. 464.

for example those given by (4.10), (4.14) and (4.19), which can be treated by the methods of this paper can be represented in one of the forms

$$M_h = A^h \prod_{i=1}^{p} \Psi(a_i, 2h) \tag{5.1}$$

and

$$M'_h = B^h \prod_{i=1}^{p} [\Psi(b_i, 2h)\, \Psi(c_i + 2h, -2h)] \tag{5.2}$$

where $c_i - b_i > 0$ $(i = 1, 2, \cdots p)$ and A and B are positive and independent of h. Thus, if U and V are statistical functions whose h-th moments are of the forms M_h and M'_h respectively, and $f(U)$ and $g(V)$ are the corresponding frequency distributions then the moment equations

$$\int_0^\infty U^h f(U)\, dU = M_h \tag{5.3}$$

and

$$\int_0^B V^h g(V) dV = M'_h \tag{5.4}$$

must be satisfied. Solutions of these equations have been found.[17] The expression (4.46) is a moment of neither the M_h nor M'_h type; however, when $N_1 = N_2 = \cdots = N_k$ it can be transformed into the M'_h form by expanding the gamma functions in $\prod_{i=1}^{t} \Psi(N - n + t + hN - i, -hN)$ by Gauss'[18] multiplication formula for gamma functions. In the case of the multiple correlation coefficient it has been found that the h-th moment is an infinite series, each term of which is, except for a constant factor, of the form (5.2).

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[17] S. S. Wilks, Biometrika, Vol. XXIV (1932) pp. 473–475.

[18] See Whittaker and Watson, Modern analysis, fourth edition (1927) p. 240.

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ON TENSOR INVARIANCE IN THE CALCULUS OF VARIATIONS

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Introduction. Morse[2] has given a tensor formulation of the problem, set up in an m-dimensional Riemannian space R^3 (with points x), of minimizing

$$\theta(\alpha) + \int_C F\left(x \,\middle|\, \frac{dx}{dt}\right) dt \tag{0.1}$$

by choice of an oriented arc C (parameterized by t) whose end points $x^{(s)}$ have the coordinates

$$x^{i(s)} = x^{i(s)}(\alpha) \qquad s = 1, 2. \tag{0.2}$$

α denotes a point of a given subspace of a $2m$-dimensional space $R^{(1)} \times R^{(2)}$, the product of two copies of R. F denotes a function positively homogeneous of degree one in its second set of arguments:

$$F(x|kr) = k\, F(x|r). \tag{0.3}$$

This paper introduces a covariant differentiation (denoted by semicolon) which applies to the above problem as follows:

$$F_{;ri} - F_{;xi} = \frac{d}{dt}F_{ri} - F_{xi} \tag{0.4}$$

$$\theta_{;\alpha h} + [F_{;ri}\, x^{i(s)}_{;\alpha h}]_1^2 = \theta_{\alpha h} + \left[F_{ri}\,\frac{\partial x^{i(s)}}{\partial \alpha^h}\right]_1^2. \tag{0.5}$$

These establish the invariant nature of the *Euler equations* (0.4) = 0, and the *transversality conditions* (0.5) = 0. For a given C satisfying (0.4) = 0 and (0.5) = 0 we have

$$\beta^*_{hk}\, u^h\, u^k + \int_C 2\omega^*(\eta \mid \eta_{;i})\, dt = \beta_{hk}\, u^h\, u^k + \int_C 2\omega\left(\eta \,\middle|\, \frac{d\eta}{dt}\right) dt \tag{0.6}$$

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[1] National Research Fellow.

[2] M. Morse, *The calculus of variations in the large*, American Mathematical Society Colloquium Publications (not yet published). I am indebted to Professor Morse for the privilege of reading the manuscript.

[3] R may be regarded as a regular manifold in the sense of Veblen-Whitehead, *The foundations of differential geometry*, Cambridge Tract (1932). Since we are not interested in details of class we merely assume that the varieties and quantities we introduce have classes sufficient for our needs.

where

$$\beta^*_{hk} = \theta_{;\alpha^h\alpha^k} + [F_{;r^i}\, x^{i(s)}_{;\alpha^h\alpha^k}]^2_1$$

$$2\omega^*(\eta \mid \sigma) = F_{;r^ir^j}\,\sigma^i\,\sigma^j + 2F_{;r^ix^j}\,\sigma^i\,\eta^j + (F_{;x^ix^j} + r^u\,F_{;r^p}\,R^p_{iju})\eta^i\,\eta^i$$

$$\beta_{hk} = \theta_{\alpha^h\alpha^k} + \left[F_{r^i}\frac{\partial^2 x^{i(s)}}{\partial\alpha^h\partial\alpha^k}\right]^2_1$$

$$2\omega(\eta \mid \pi) = F_{r^ir^j}\,\pi^i\,\pi^j + 2F_{r^ix^j}\,\pi^i\,\eta^j + F_{x^ix^j}\,\eta^i\,\eta^j,$$

provided

$$\eta^{i(s)} = \frac{\partial x^{i(s)}}{\partial\alpha^h}\,u^h. \tag{0.7}$$

Moreover

$$\omega^*_{;\sigma^i\,t} - \omega^*_{;\eta^i} = \frac{d}{dt}\omega_{\pi^i} - \omega_{\eta^i} \tag{0.8}$$

$$\beta^*_{hk}\,u^k + \left[\omega^*_{;\sigma^i}\frac{\partial x^{i(s)}}{\partial\alpha^h}\right]^2_1 = \beta_{hk}\,u^k + \left[\omega_{\pi^i}\frac{\partial x^{i(s)}}{\partial\alpha^h}\right]^2_1. \tag{0.9}$$

These establish the invariant nature of an *accessory problem* based on the *second variation* (0.6), the *secondary end conditions* (0.7), the *Jacobi equations* (0.8) = 0, and the *secondary transversality conditions* (0.9) = 0.

In §1 we develop our covariant differentiation by itself as an interesting extension of tensor analysis. In the later sections we apply it to multiple integral problems in the calculus of variations.

1. **Covariant differentiation.**[4] Let r denote a mixed quantity whose components transform according to the rules

$$r^{i'}_\lambda = r^i_\lambda\frac{\partial x^{i'}}{\partial x^i}, \qquad r^i_{\lambda'} = r^i_\lambda\,k^\lambda_{\lambda'}, \qquad k \equiv |k^\lambda_{\lambda'}| > 0. \tag{1.1}$$

Let Q denote a quantity, dependent on x and r, which is simply covariant in x and a simply contravariant "direct" density in r:

$$Q^\lambda_{i'} = Q^\lambda_i\frac{\partial x^i}{\partial x^{i'}}, \qquad Q^{\lambda'}_i = k\,Q^\nu_i\,k^{\lambda'}_\nu \qquad (k^\lambda_{\lambda'}\,k^{\lambda'}_\nu = \delta^\lambda_\nu). \tag{1.2}$$

("Direct" refers to the restriction $k > 0$.)

To avoid complicated subscripts we write the partial derivatives of Q with respect to x^j and r^j_μ as

$$Q^\lambda_{i,j} \quad \text{and} \quad Q^{\lambda\,\mu}_{i,j}.$$

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[4] For tensor terminology see L. P. Eisenhart, *Riemannian Geometry*, Princeton (1926); O. Veblen, *Invariants of quadratic differential forms*, Cambridge Tract (1927).

Multiplying the second part of (1.2) by $k^{\lambda}_{\lambda'}$, summing, and differentiating with respect to $k^{\mu}_{\mu'}$ we get

$$Q^{\lambda'\mu'}_{i,j}\, r^{j}_{\mu}\, k^{\lambda}_{\lambda'} + Q^{\mu'}_{i}\, \delta^{\lambda}_{\mu} = k\, Q^{\lambda}_{i}\, k^{\mu'}_{\mu}.$$

With $k^{\lambda}_{\lambda'}$ a Kronecker delta this becomes

$$Q^{\lambda\,\mu}_{i,j}\, r^{j}_{\nu} + Q^{\mu}_{i}\, \delta^{\lambda}_{\nu} = Q^{\lambda}_{i}\, \delta^{\mu}_{\nu}. \tag{1.3}$$

By differentiating (1.2) with respect to r^{j}_{μ} and x^{j} we get

$$\begin{cases} Q^{\lambda\,\mu}_{i,j'}\, \dfrac{\partial x^{j'}}{\partial x^{j}} = Q^{\lambda\,\mu}_{i,j}\, \dfrac{\partial x^{i}}{\partial x^{i'}}, \qquad Q^{\lambda'\mu'}_{i,j}\, k^{\mu}_{\mu'} = k\, Q^{\nu\,\mu}_{i,j}\, k^{\lambda'}_{\nu} \\ Q^{\lambda}_{i',j'}\, \dfrac{\partial x^{j'}}{\partial x^{j}} + Q^{\lambda}_{i',j'}\, \dfrac{\partial^{2} x^{j'}}{\partial x^{i} \partial x^{j}}\, r^{i}_{\mu} = Q^{\lambda}_{i,j}\, \dfrac{\partial x^{i}}{\partial x^{i'}}, \qquad Q^{\lambda'}_{i,j} = k\, Q^{\nu}_{i,j}\, k^{\lambda'}_{\nu}. \end{cases} \tag{1.4}$$

We define the covariant derivatives of Q by the rules

$$\begin{cases} Q^{\lambda\,\mu}_{i;j} = Q^{\lambda\,\mu}_{i,j} \\ Q^{\lambda}_{i;j} = Q^{\lambda}_{i,j} - Q^{\lambda}_{p}\, \Gamma^{p}_{i\,j} - Q^{\lambda\,\mu}_{i,p}\, \Gamma^{p}_{j\,q}\, r^{q}_{\mu}. \end{cases} \tag{1.5}$$

(It is indifferent here whether the Γ's are Christoffel symbols or the components of a more general connection.) Using (1.4) it may be readily verified that these covariant derivatives have the tensor character

$$Q^{\lambda\,\mu}_{i';j'} = Q^{\lambda\,\mu}_{i;j}\, \frac{\partial x^{i}}{\partial x^{i'}}\, \frac{\partial x^{j}}{\partial x^{j'}}, \qquad Q^{\lambda'\mu'}_{i;j} = k\, Q^{\lambda\,\mu}_{i;j}\, k^{\lambda'}_{\lambda}\, k^{\mu'}_{\mu}$$

$$Q^{\lambda}_{i';j'} = Q^{\lambda}_{i;j}\, \frac{\partial x^{i}}{\partial x^{i'}}\, \frac{\partial x^{j}}{\partial x^{j'}}, \qquad Q^{\lambda'}_{i;j} = k\, Q^{\lambda}_{i;j}\, k^{\lambda'}_{\lambda}.$$

Now let the components of Q have the arguments

$$x^{i} = x^{i}(t) \quad \text{and} \quad r^{i}_{\lambda} = \frac{\partial x^{i}}{\partial t^{\lambda}}.$$

t denotes a point in an oriented l-dimensional subspace C. Transformations of coordinates (parameters) in this subspace are required to preserve the orientation, i.e. $\left| \partial t^{\lambda} / \partial t^{\lambda'} \right| > 0$. We define the covariant derivatives of Q in the subspace by the rule

$$Q^{\lambda}_{i;\nu} = \frac{\partial}{\partial t^{\nu}} Q^{\lambda}_{i} - Q^{\lambda}_{p}\, \Gamma^{p}_{i\,q}\, \frac{\partial x^{q}}{\partial t^{\nu}} + Q^{\mu}_{i}\, \Gamma^{\lambda}_{\mu\,\nu} - Q^{\lambda}_{i}\, \Gamma^{\mu}_{\mu\,\nu}. \tag{1.6}$$

($\Gamma^{\lambda}_{\mu\,\nu}$ denote Γ's in the subspace.) Using (1.3) and the covariant derivatives

$$x^{i}_{;\mu\,\nu} = \frac{\partial^{2} x^{i}}{\partial t^{\mu} \partial t^{\nu}} + \frac{\partial x^{p}}{\partial t^{\mu}}\, \Gamma^{i}_{p\,q}\, \frac{\partial x^{q}}{\partial t^{\lambda}} - \frac{\partial x^{i}}{\partial t^{\lambda}}\, \Gamma^{\lambda}_{\mu\,\nu} \tag{1.7}$$

we see that (1.6) obeys the "function of a function" rule

$$Q^{\lambda}_{i;\nu} = Q^{\lambda}_{i;j}\, \frac{\partial x^{j}}{\partial t^{\nu}} + Q^{\lambda\,\mu}_{i;j}\, x^{j}_{;\mu\,\nu}. \tag{1.8}$$

Hence $Q_{;}$ has the tensor character

$$Q^{\lambda}_{i';\nu} = Q^{\lambda}_{i;\nu}\frac{\partial x^{i}}{\partial x^{i'}}, \qquad Q^{\lambda'}_{i;\nu'} = \left|\frac{\partial t}{\partial t'}\right| Q^{\lambda}_{i;\nu}\frac{\partial t^{\lambda'}}{\partial t^{\lambda}}\frac{\partial t^{\nu}}{\partial t^{\nu'}}.$$

By contraction (1.6) becomes the divergence

$$Q^{\lambda}_{i;\lambda} = \frac{\partial}{\partial t^{\lambda}} Q^{\lambda}_{i} - Q^{\lambda}_{p}\,\Gamma^{p}_{iq}\frac{\partial x^{q}}{\partial t^{\lambda}}. \tag{1.9}$$

The rules given above apply with obvious modifications to a Q having components with an arbitrary number of indices, covariant or contravariant, and with arbitrary weight. In particular they apply to a Q which is scalar. They also apply to a Q which depends on more than one quantity of the r type.

If the subspace C is a curve, i.e. if $l = 1$, Greek indices may be dispensed with, provided a more explicit means of indicating differentiation is used—say that in our introduction. Attached to the curve there is just one Γ; in the metric case it has the value $(d^2s/dt^2) \div (ds/dt)$, s being the arc length along the curve.

2. **Integral of a divergence.** Let

$$\int_C f\,dt \tag{2.1}$$

symbolize an l-fold integral taken over an oriented l-dimensional *field*[5] of integration C (parameterized by t) immersed in the underlying x-space. f denotes a scalar (direct) density in t; dt denotes a "volume element" $dt^1dt^2\cdots dt^l$. Let γ (parameterized by τ) denote the boundary of C. If f is a divergence

$$\phi^{\lambda}{}_{;\lambda} = \frac{\partial}{\partial t^{\lambda}}\phi^{\lambda}$$

the integral (2.1) is equivalent to the $(l-1)$-fold integral over γ of the component of ϕ normal to γ

$$\int_C \frac{\partial}{\partial t^{\lambda}}\phi^{\lambda}\,dt = \int_{\gamma}\xi_{\lambda}\,\phi^{\lambda}\,d\tau. \tag{2.2}$$

ξ is a covariant vector of weight -1 in t and a scalar density in τ. The components of ξ are the principal minors of $\| \partial t^{\lambda}/\partial\tau^{\rho} \|$:

$$\xi_{\lambda} = \epsilon_{\lambda\mu\cdots\nu}\frac{\partial t^{\mu}}{\partial\tau^{1}}\cdots\frac{\partial t^{\nu}}{\partial\tau^{l-1}}.$$

Formula (2.2) generalizes the well-known theorems ($l = 2, 3$) which bear the name of Green (or Gauss). If C is an arc bounded by $\gamma = \tau_2 - \tau_1$, (2.2) becomes

$$\int_C\left(\frac{d}{dt}\phi\right)dt = [\phi]^{\tau_2}_{\tau_1}.$$

[5] Cf. G. de Rham, J. de Math. IX s., **10** (1931), p. 166.

3. **Problem with fixed boundary.** Turning to the calculus of variations we take an integral of arbitrary multiplicity

$$\int_C F\left(x \,\middle|\, \frac{\partial x}{\partial t}\right) dt. \tag{3.1}$$

We admit only C's bounded by a fixed γ. $F(x \mid r)$ denotes a quantity which is a scalar in x and a scalar direct density in r. Accordingly, the integral (3.1) is invariant with respect to coordinate transformations of x and orientation-preserving coordinate (parameter) transformations of t.

The first variation of (3.1) is

$$\int_C \left(F_{,i}^{\lambda} \frac{\partial \eta^i}{\partial t^\lambda} + F_{,i}\,\eta^i\right) dt = \int_\gamma \xi_\lambda F_{,i}^{\lambda}\,\eta^i\, d\tau - \int_C \left(\frac{\partial}{\partial t^\lambda} F_{,i}^{\lambda} - F_{,i}\right)\eta^i\, dt \tag{3.2}$$

for, using (2.2),

$$\int_C \frac{\partial}{\partial t^\lambda}\left(F_{,i}^{\lambda}\,\eta^i\right) dt = \int_\gamma \xi_\lambda F_{,i}^{\lambda}\,\eta^i\, d\tau.$$

η denotes the first variation of x on C; it is a contravariant vector such that

$$\eta^i = 0 \qquad \text{on } \gamma. \tag{3.3}$$

The vanishing of (3.2), reduced by (3.3), leads to the *Euler equations*

$$\frac{\partial}{\partial t^\lambda} F_{,i}^{\lambda} - F_{,i} = 0. \tag{3.4}$$

These are tensor equations, for

$$F_{;i\lambda}^{\lambda} - F_{;i} = \frac{\partial}{\partial t^\lambda} F_{,i}^{\lambda} - F_{,i} \tag{3.5}$$

as may be seen by subtracting (cf. (1.5) and (1.9))

$$F_{;i} = F_{,i} - F_{,p}^{\lambda}\,\Gamma_{iq}^{p}\, r_\lambda^q$$

from

$$F_{;i\lambda}^{\lambda} = \frac{\partial}{\partial t^\lambda} F_{,i}^{\lambda} - F_{,p}^{\lambda}\,\Gamma_{iq}^{p}\,\frac{\partial x^q}{\partial t^\lambda}.$$

4. **Second variation.** The second variation of (3.1), calculated for a given C, is

$$\int_C \left(F_{,i}^{\lambda} \frac{\partial \zeta^i}{\partial t^\lambda} + F_{,i}\,\zeta^i\right) dt + \int_C 2\omega\left(\eta \,\middle|\, \frac{\partial \eta}{\partial t}\right) dt \tag{4.1}$$

where

$$2\omega(\eta|\pi) = F_{,ij}^{\lambda\mu}\,\pi_\lambda^i\,\pi_\mu^j + 2F_{,ij}^{\lambda}\,\pi_\lambda^i\,\eta^j + F_{;ij}\,\eta^i\,\eta^j.$$

ζ denotes the second variation of x on C. It is such that

$$\zeta^i = 0. \qquad \text{on } \gamma. \tag{4.2}$$

ζ is not a contravariant vector but

$$\zeta^i + \eta^p \, \Gamma^i_{pq} \, \eta^q$$

are the components of a contravariant vector. Similarly π is not a quantity of the r type (cf. (1.1)), but σ is, where

$$\sigma^i_\lambda = \pi^i_\lambda + \eta^p \, \Gamma^i_{pq} \, r^q_\lambda. \tag{4.3}$$

Like (3.2)

$$\int_C \left(F_{,i}^{\,\lambda} \frac{\partial \zeta^i}{\partial t^\lambda} + F_{,i} \, \zeta^i \right) dt = \int_\gamma \xi_\lambda \, F_{,i}^{\,\lambda} \, \zeta^i \, d\tau - \int \left(\frac{\partial}{\partial t^\lambda} F_{,i}^{\,\lambda} - F_{,i} \right) \zeta^i \, dt. \tag{4.4}$$

We assume that C satisfies the Euler equations (3.4). Then the second term on the right of (4.4) vanishes. Due to (4.2) the first term vanishes. Hence (4.1) reduces to

$$\int_C 2\omega \left(\eta \,\middle|\, \frac{\partial \eta}{\partial t} \right) dt. \tag{4.5}$$

In place of (4.5) we consider

$$\int_C 2\omega^*(\eta|\eta;) \, dt \tag{4.6}$$

where

$$2\omega^*(\eta|\sigma) = F_{;i\,j}^{\,\lambda\mu} \, \sigma^i_\lambda \, \sigma^j_\mu + 2F_{;i\,j}^{\,\lambda} \, \sigma^i_\lambda \, \eta^j + (F_{;i\,j} + r^u_\lambda \, F_{;p}^{\,\lambda} \, R^p_{iju}) \eta^i \, \eta^j.$$

The components of the curvature tensor appearing in the coefficients of $\eta^i \eta^j$ in $2\omega^*(\eta \mid \sigma)$ serve to render these coefficients symmetric in the indices i, j. $2\omega^*(\eta \mid \eta;)$ is a scalar in x and a scalar direct density in t. Hence (4.6) is invariant.

By (4.3) and our rules of covariant differentiation (cf. (1.5))

$$\left\{ \begin{aligned}
F_{;i\,j}^{\,\lambda\mu} \, \sigma^i_\lambda \, \sigma^j_\mu &= F_{,i\,j}^{\,\lambda\mu} \, (\pi^i_\lambda + \Gamma^i_{pu} \, r^u_\lambda \, \eta^p) \, (\pi^j_\mu + \Gamma^j_{qv} \, r^v_\mu \, \eta^q) \\
&= F_{,i\,j}^{\,\lambda\mu} \, \pi^i_\lambda \, \pi^j_\mu + 2F_{,i\,q}^{\,\lambda\mu} \, \Gamma^q_{jv} \, r^v_\mu \, \pi^i_\lambda \, \eta^j + F_{,p\,q}^{\,\lambda\mu} \, \Gamma^p_{iu} \, \Gamma^q_{jv} \, r^u_\lambda \, r^v_\mu \, \eta^i \, \eta^j \\
F_{;i\,j}^{\,\lambda} \, \sigma^i_\lambda \, \eta^j &= (F_{,i\,j}^{\,\lambda} - F_{,p}^{\,\lambda} \, \Gamma^p_{i\,j} - F_{,i\,q}^{\,\lambda\mu} \, \Gamma^q_{jv} \, r^v_\mu) \, (\pi^i_\lambda + \Gamma^i_{wu} \, r^u_\lambda \, \eta^w) \, \eta^j \\
&= (F_{,i\,j}^{\,\lambda} - F_{,p}^{\,\lambda} \, \Gamma^p_{i\,j} - F_{,i\,q}^{\,\lambda\mu} \, \Gamma^q_{jv} \, r^v_\mu) \, \pi^i_\lambda \, \eta^j \\
&\quad + (F_{,p\,j}^{\,\lambda} \, \Gamma^p_{iu} \, r^u_\lambda - F_{,p}^{\,\lambda} \, \Gamma^p_{w\,j} \, \Gamma^w_{iu} \, r^u_\lambda - F_{,p\,q}^{\,\lambda\mu} \, \Gamma^p_{iu} \, \Gamma^q_{jv} \, r^u_\lambda \, r^v_\mu) \, \eta^i \, \eta^j \\
F_{;ij} \, \eta^i \, \eta^j &= \Big\{ F_{,ij} - F_{,p\,j}^{\,\lambda} \, \Gamma^p_{iu} \, r^u_\lambda - F_{,p}^{\,\lambda} \left(\frac{\partial}{\partial x^j} \Gamma^p_{iu} \right) r^u_\lambda - F_{,p} \, \Gamma^p_{i\,j} \\
&\quad + F_{,p}^{\,\lambda} \, \Gamma^p_{uw} \, \Gamma^w_{i\,j} \, r^u_\lambda - F_{,p\,i}^{\,\lambda} \, \Gamma^p_{ju} \, r^u_\lambda + F_{,p\,q}^{\,\lambda\mu} \, \Gamma^p_{iu} \, \Gamma^q_{jv} \, r^u_\lambda \, r^v_\mu \\
&\quad + F_{,p}^{\,\lambda} \, \Gamma^p_{iw} \, \Gamma^w_{ju} \, r^u_\lambda \Big\} \eta^i \, \eta^j.
\end{aligned} \right. \tag{4.7}$$

To these we add

$$(4.8)\quad r^u_\lambda F^{\ \lambda}_{,p} R^p_{iju}\,\eta^i\,\eta^j = r^u_\lambda F^{\ \lambda}_{,p}\left(\frac{\partial}{\partial x^j}\Gamma^p_{iu} - \frac{\partial}{\partial x^u}\Gamma^p_{ij} + \Gamma^p_{jw}\,\Gamma^w_{iu} - \Gamma^p_{uw}\,\Gamma^w_{ij}\right)\eta^i\,\eta^j$$

to get that

$$2\omega^*(\eta\,|\,\sigma) = 2\omega\,(\eta\,|\,\pi) - \left\{2F^{\ \lambda}_{,p}\,\Gamma^p_{ij}\,\pi^i_\lambda\,\eta^j + F_{,p}\,\Gamma^p_{ij}\,\eta^i\,\eta^j + F^{\ \lambda}_{,p}\,r^u_\lambda\left(\frac{\partial}{\partial x^u}\Gamma^p_{ij}\right)\eta^i\,\eta^j\right\}.$$

On C the bracket in the last equation reduces to

$$\frac{\partial}{\partial t^\lambda}\,(F^{\ \lambda}_{,p}\,\Gamma^p_{ij}\,\eta^i\,\eta^j)$$

since C satisfies the Euler equations (3.4). So integrating, with use of (2.2), we have

$$(4.9)\qquad \int_C 2\omega^*(\eta\,|\,\eta_i)\,dt = \int_C 2\omega\left(\eta\,\middle|\,\frac{\partial\eta}{\partial t}\right)dt - \int_\gamma \xi_\lambda\,F^{\ \lambda}_{,p}\,\Gamma^p_{ij}\,\eta^i\,\eta^j\,d\tau.$$

By (3.3) the boundary integral vanishes. Hence (4.6) and (4.5) are equivalent. This proves the invariance of (4.5).

5. **Jacobi equations.** The Euler equations based on (4.5) are the *Jacobi equations*

$$(5.1)\qquad \frac{\partial}{\partial t^\mu}\omega^{\ \mu}_{,j} - \omega_{,j} = 0.$$

To establish their invariant nature we show that

$$(5.2)\qquad \omega^{*\ \mu}_{;\ j\,\mu} - \omega^*_{;j} = \frac{\partial}{\partial t^\mu}\omega^{\ \mu}_{,j} - \omega_{,j}.$$

The ordinary derivatives of $2\omega^*$ with respect to σ^i_λ and η^j are covariant derivatives (cf. first part of (1.5)). So

$$(5.3)\quad \begin{cases} \omega^{*\lambda}_{;\ i} = F^{\lambda\,\mu}_{;\,i\,j}\,\sigma^j_\mu + F^{\ \lambda}_{;\,i\,j}\,\eta^j \\ \qquad = F^{\lambda\,\mu}_{,\,i\,j}(\pi^j_\mu + \Gamma^j_{qv}\,r^v_\mu\,\eta^q) + (F^{\ \lambda}_{,\,i\,j} - F^{\lambda\,\mu}_{,\,i\,q}\,\Gamma^q_{jv}\,r^v_\mu - F_{,p}\,\Gamma^p_{ij})\,\eta^j \\ \qquad = F^{\lambda\,\mu}_{,\,i\,j}\,\pi^j_\mu + F^{\ \lambda}_{,\,i\,j}\,\eta^j - F^{\ \lambda}_{,p}\,\Gamma^p_{ij}\,\eta^j. \end{cases}$$

Changing indices and evaluating on C we have

$$\omega^{*\ \mu}_{;\ j} = F^{\lambda\,\mu}_{,\,i\,j}\,\frac{\partial\eta^i}{\partial t^\lambda} + F^{\ \mu}_{,\,i\,j}\,\eta^i - F^{\ \mu}_{,p}\,\Gamma^p_{ij}\,\eta^i,$$

the divergence of which is (cf. (1.9))

$$(5.4)\quad \begin{cases} \omega^{*\ \mu}_{;\ j\,\mu} = \dfrac{\partial}{\partial t^\mu}\left(F^{\lambda\,\mu}_{,\,i\,j}\,\dfrac{\partial\eta^i}{\partial t^\lambda} + F^{\ \mu}_{,\,i\,j}\,\eta^i\right) - \dfrac{\partial}{\partial t^\mu}\,(F^{\ \mu}_{,p}\,\Gamma^p_{ij}\,\eta^i) \\ \qquad - F^{\lambda\,\mu}_{,\,i\,q}\,\Gamma^q_{jv}\,\dfrac{\partial x^v}{\partial t^\mu}\,\dfrac{\partial\eta^i}{\partial t^\lambda} - F^{\ \lambda}_{,p\,i}\,\Gamma^p_{ju}\,\dfrac{\partial x^u}{\partial t^\lambda}\,\eta^i + F^{\ \lambda}_{,p}\,\Gamma^p_{iw}\,\Gamma^w_{ju}\,\dfrac{\partial x^u}{\partial t^\lambda}\,\eta^i. \end{cases}$$

By using (4.7) we see that on C

$$(5.5)\quad \begin{cases} \omega^*_{;j} = F^{\ \lambda}_{;i\ j}\ \eta^{;i}_{;\lambda} + (F_{;ij} + r^u_\lambda\, F^{\ \lambda}_{;p}\, R^p_{i\,j\,u})\,\eta^i \\ \qquad = F^{\ \lambda}_{,i\ j}\,\dfrac{\partial \eta^i}{\partial t^\lambda} + F_{,ij}\,\eta^i - \dfrac{\partial}{\partial t^\lambda}\,(F^{\ \lambda}_{,p}\,\Gamma^p_{i\,j}\,\eta^i) \\ \qquad - F^{\ \lambda\ \mu}_{,i\ q}\,\Gamma^q_{j\,v}\,\dfrac{\partial x^v}{\partial t^\mu}\,\dfrac{\partial \eta^i}{\partial t^\lambda} - F^{\ \lambda}_{,p\ i}\,\Gamma^p_{j\,u}\,\dfrac{\partial x^u}{\partial t^\lambda}\,\eta^i + F^{\ \lambda}_{,p}\,\Gamma^p_{i\,w}\,\Gamma^w_{j\,u}\,\dfrac{\partial x^u}{\partial t^\lambda}\,\eta^i \end{cases}$$

since C satisfies the Euler equations (3.4). Therefore, subtracting (5.5) from (5.4), we get (5.2).

6. **Problem with variable boundary.** Instead of (3.1) we consider

$$(6.1)\qquad \int_\gamma \Theta(z \mid t)\, d\tau + \int_C F\left(x \,\middle|\, \frac{\partial x}{\partial t}\right) dt$$

and instead of keeping γ fixed we set it free to vary over a given subspace of our x-space. z denotes a point of this subspace. Θ denotes a scalar in z and a scalar (direct) density in τ. Hence (6.1) is invariant.

Instead of (3.3) we now have

$$(6.2)\qquad \eta^i = \frac{\partial x^i}{\partial z^a}\, v^a \qquad \text{on } \gamma.$$

v, the first variation of z on γ, is a contravariant vector. Using (3.2) and (6.2) we have that the first variation of (6.1) is

$$(6.3)\qquad \int_\gamma \left(\Theta_{,a} + \xi_\lambda\, F^{\ \lambda}_{,i}\,\frac{\partial x^i}{\partial z^a}\right) v^a\, d\tau - \int_C \left(\frac{\partial}{\partial t^\lambda}\, F^{\ \lambda}_{,i} - F_{,i}\right)\eta^i\, dt.$$

Let (6.3) vanish. The vanishing of the second integral leads to the Euler equations (3.4). The vanishing of the first leads to the *transversality conditions*

$$(6.4)\qquad \Theta_{,a} + \xi_\lambda\, F^{\ \lambda}_{,i}\,\frac{\partial x^i}{\partial z^a} = 0.$$

These are tensor equations since the partial derivatives in (6.4) are covariant derivatives.

Instead of (4.2) we now have

$$(6.5)\qquad \zeta^i = \frac{\partial x^i}{\partial z^a}\, w^a + \frac{\partial^2 x^i}{\partial z^a \partial z^b}\, v^a\, v^b \qquad \text{on } \gamma.$$

w is the second variation of z on γ. Using (4.1), (4.4), and (6.5) we get that the second variation of (6.1), calculated for a given C, is

$$\int_\gamma \left(\Theta_{,a} + \xi_\lambda\, F^{\ \lambda}_{,i}\,\frac{\partial x^i}{\partial z^a}\right) w^a\, d\tau - \int_C \left(\frac{\partial}{\partial t^\lambda}\, F^{\ \lambda}_{,i} - F_{,i}\right)\zeta^i\, dt + \int_\gamma B_{ab}\, v^a\, v^b\, d\tau$$
$$+ \int_C 2\omega\left(\eta \,\middle|\, \frac{\partial \eta}{\partial t}\right) dt$$

where

$$B_{ab} = \Theta_{,ab} + \xi_\lambda F_{,i}^{\lambda} \frac{\partial^2 x^i}{\partial z^a \partial z^b}.$$

We assume that C satisfies the Euler equations (3.4) and the transversality conditions (6.4). The second variation then reduces to

$$\int_\gamma B_{ab}\, v^a\, v^b\, d\tau + \int_C 2\omega\left(\eta \,\middle|\, \frac{\partial \eta}{\partial t}\right) dt. \tag{6.6}$$

In place of (6.6) we consider

$$\int_\gamma B^*_{a\,b}\, v^a\, v^b\, d\tau + \int_C 2\omega^*(\eta \mid \eta;)\, dt \tag{6.7}$$

where

$$B^*_{a\,b} = \Theta_{;ab} + \xi_\lambda F_{;i}^{\lambda}\, x^i_{;a\,b}.$$

B^* is doubly covariant in z and a scalar direct density in τ. Hence (6.7) is invariant. By our rules of covariant differentiation (cf. (1.7))

$$\begin{cases} B^*_{a\,b} = \Theta_{,ab} - \Theta_{,c}\, \Gamma^c_{a\,b} + \xi_\lambda F_{,i}^{\lambda} \left(\dfrac{\partial^2 x^i}{\partial z^a \partial z^b} + \dfrac{\partial x^p}{\partial z^a}\, \Gamma^i_{p\,q}\, \dfrac{\partial x^q}{\partial z^b} - \dfrac{\partial x^i}{\partial z^c}\, \Gamma^c_{a\,b} \right) \\ \qquad = B_{ab} + \xi_\lambda F_{,p}^{\lambda}\, \Gamma^p_{i\,j}\, \dfrac{\partial x^i}{\partial z^a} \dfrac{\partial x^j}{\partial z^b}. \end{cases} \tag{6.8}$$

since C satisfies the transversality conditions (6.4). Now, using (4.9) and (6.2), as well as (6.8), we see that (6.7) and (6.6) are equivalent. This proves that (6.6) is invariant.

In addition to the Jacobi equations there are now *secondary transversality conditions*

$$B_{ab}\, v^b + \xi_\lambda\, \omega_{,i}^{\lambda} \frac{\partial x^i}{\partial z^a} = 0. \tag{6.9}$$

These are tensor equations since (5.3), (6.2), and (6.8) unite to show that

$$B^*_{a\,b}\, v^b + \xi_\lambda\, \omega^{*\lambda}_{;\,i} \frac{\partial x^i}{\partial z^a}$$

reduces to the left side of (6.9).

7. **Remark.** The problem of Morse treated in our introduction is more general than the simple integral case of the problem considered in the last section. The former reduces to the latter if the α-space (cf. (0.2) f.) is the product of two subspaces of our x-space, and if θ is expressible as the difference of functions defined over the two subspaces. Despite this added generality the formulas

given in the introduction are quite analogous to those we have been developing in the last few sections. To induce a metric in the α-space we need only assume that the square of the linear element in $R^{(1)} \times R^{(2)}$ is just the sum of the squares of the linear elements in $R^{(1)}$ and $R^{(2)}$. Then

$$x^{i(s)}_{;\alpha^h \alpha^k} = \frac{\partial^2 x^{i(s)}}{\partial \alpha^h \partial \alpha^k} + \frac{\partial x^{p(s)}}{\partial \alpha^h} \Gamma^i_{pq} \frac{\partial x^{q(s)}}{\partial \alpha^k} - \frac{\partial x^{i(s)}}{\partial \alpha^g} \Gamma^g_{hk}.$$

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HAUSDORFF GROUPOIDS

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1. **Introduction.** Associated with any Hausdorff space is a topological invariant, namely, the Hausdorff space of its $(1 \to 1)$ transformations into itself, under a suitable definition of "neighborhood." If the points of the primitive Hausdorff space are isolated, this definition is equivalent to Kneser's definition[1] of continuity, and the resulting transformation-space may be known as a "Hausdorff groupoid."

Under further special conditions, this Hausdorff groupoid has special metric properties. An immediate application of the general discussion is the fact that the group of any algebraic extension of an enumerable commutative ring is a Cantor set, an M-group in the sense of van Dantzig,[2] and has a mass-function in the sense of Caratheodory.[3]

By connection with this result, we investigate the order of the group of automorphisms of the complex number system, the general question of automorphic extension, and the lattice theory of Galois extensions.[4]

2. **Groupoid.** By a "groupoid" we mean any system $\mathfrak{G}$ whose elements satisfy the first three group axioms:[5]

G1: Every ordered pair of elements α and β of $\mathfrak{G}$ determines a product element $\alpha\beta$ in $\mathfrak{G}$.

G2: $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for any α, β, and γ in $\mathfrak{G}$.

G3: $\mathfrak{G}$ contains an element $\underline{1}$ satisfying $\alpha\underline{1} = \underline{1}\alpha = \alpha$ for any α in $\mathfrak{G}$.

Let α_1 be a fixed, and γ a variable element of the groupoid $\mathfrak{G}$. The transformation α_1: $\gamma \to \gamma\alpha_1$ is evidently a $(1 \to 1)$ transformation of the elements of $\mathfrak{G}$, and the transformation $\alpha_1\alpha_2$: $\gamma \to \gamma(\alpha_1\alpha_2)$ is identical by (G2) with the sequence $\gamma \to (\gamma\alpha_1) \to (\gamma\alpha_1)\alpha_2$. Since also only α_1 carries $\underline{1}$ into α_1, we see that distinct elements perform distinct transformations on $\mathfrak{G}$. That is, to any groupoid corresponds a (1,1) isomorphic set of $(1 \to 1)$ transformations of a class into itself.

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[1] Math. Zeits. 25 (1926), p. 362.

[2] Math. Ann. 107 (1933), 587–626.

[3] "Vorlesungen über reelle Funktionen," Berlin, 1927, p. 238.

[4] Cf. the author's "On the combination of subalgebras," Proc. Camb. Phil. Soc., 29 (1933), 441–464.

[5] Cf. W. Specht, "Eine Verallgemeinerung der Permutationsgruppen," Math. Zeits., 37 (1933), 321–41. Not to be confused with the "gruppoid" of H. Brandt, in "Über eine Verallgemeinerung des Gruppenbegriffs," Math. Ann., 96 (1926), 360–66.

The obviousness of the converse to this shows

THEOREM 1: *Groupoid algebra is the algebra of those sets of* $(1 \to 1)$ *transformations which contain the identity, and the product of any two transformations of the set.*

By a "subgroupoid" of any groupoid $\mathfrak{G}$, is meant any set of elements of $\mathfrak{G}$ which itself constitutes a groupoid.

3. Transformations of Hausdorff space. We shall be dealing almost invariably with Hausdorff spaces in the conventional sense, namely,[6] with classes $\mathfrak{H}$ of points and "neighborhoods" (special sets of points of $\mathfrak{H}$) satisfying

H1: Every point x of $\mathfrak{H}$ "has" at least one neighborhood V_x, and every V_x contains x.

H2: To any V_x and V'_x corresponds a $V''_x \subset V_x \cdot V'_x$.

H3: To any $y \subset V_x$ corresponds a $V_y \subset V_x$.

H4: To any $x \neq y$ correspond V_x and V_y without common point.

Consider the class of $(1 \to 1)$ transformations τ of any Hausdorff space $\mathfrak{H}$ into itself. We can evidently regard the τ as points. In addition, we can define "neighborhoods" as follows,

HT1: The neighborhoods N_α of any fixed α shall consist of those transformations τ carrying preassigned finite sets $x_1, \cdots, x_n$ into preassigned neighborhoods $V_{\alpha(x_1)}, \cdots, V_{\alpha(x_n)}$ of $\alpha(x_1), \cdots, \alpha(x_n)$.

In terms of these, we can assert successively

H1: $N_\alpha \supset \alpha$, since by H1$\mathfrak{H}$ any $V_{\alpha(x_i)} \subset \alpha(x_i)$.

H2: Let N_α and N'_α be given, defined by requiring $\tau(x_h) \subset V_{\alpha(x_h)}$ for $h = 1, \cdots, m$ and $\tau(y_k) \subset V'_{\alpha(y_k)}$ for $k = 1, \cdots, n$ respectively. Let $z_1, \cdots, z_p$ denote the point-set sum of the x_h and the y_k; evidently $p \leqq m + n$. If z_i is both an x_h and a y_k, choose $V_{\alpha(z_i)} \subset V_{\alpha(x_h)} \cdot V'_{\alpha(y_k)}$; this is possible by H2$\mathfrak{H}$. Otherwise set $V_{\alpha(z_i)} = V_{\alpha(x_h)}$ or $V_{\alpha(z_i)} = V'_{\alpha(y_k)}$ according as $z_i = x_h$ or $z_i = y_k$. The neighborhood N''_α defined as in HT1 from these satisfies $N''_\alpha \subset N_\alpha \cdot N'_\alpha$.

H3: Suppose $\beta \subset N_\alpha$; that is, $\beta(x_i) \subset V_{\alpha(x_i)}$ for $i = , \cdots 1, n$. We can, by H3$\mathfrak{H}$, choose $V_{\beta(x_i)} \subset V_{\alpha(x_i)}$; moreover the x_i and the requirement $\tau(x_i) \subset V_{\beta(x)}$ define a $N_\beta \subset N_\alpha$.

H4: Suppose $\alpha \neq \beta$. For some x, $\alpha(x) \neq \beta(x)$. Hence, by H4$\mathfrak{H}$, there exist $V_{\alpha(x)}$ and $V_{\beta(x)}$ without common point. The N_α and N_β defined by $V_{\alpha(x)}$ and $V_{\beta(x)}$ are therefore also disjoint, as the possibilities $\tau(x) \subset V_{\alpha(x)}$ and $\tau(x) \subset V_{\beta(x)}$ are mutually exclusive.

Summarizing the results of the last four paragraphs, we get

THEOREM 2: *The* $(1 \to 1)$ *transformations of any Hausdorff space* $\mathfrak{H}$ *into itself are a groupoid* $\mathfrak{G}(\mathfrak{H})$ *homeomorphic under* HT1 *with a second Hausdorff space.*

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[6] F. Hausdorff, "Grundzüge der Mengenlehre," Leipzig, 1914, p. 213. By Hn$\mathfrak{H}$ will be meant "Hn applied to $\mathfrak{H}$."

4. **Regularity.** A topological space is said[7] to be "regular" if and only if to every point x and every closed set $\bar{S}$ not containing x correspond disjoint open sets $U \supset x$ and $V \supset \bar{S}$. We can supplement Theorem 2 by

THEOREM 3: *In Theorem* 2, *if* $\mathfrak{H}$ *is regular, then so is* $\mathfrak{G}(\mathfrak{H})$.

Suppose an element α of $\mathfrak{G}$, and a closed set $\bar{G}$ of $\mathfrak{G}$ not containing α. By definition of closure, N_α exists excluding $\bar{G}$; i.e., there exist $x_1, \cdots, x_n$ and neighborhoods $V_{\alpha(x1)}, \cdots, V_{\alpha(x_n)}$, such that to $\gamma \subset \bar{G}$ corresponds an x_i for which $\gamma(x_i) \notin V_{\alpha(x_i)}$.

Now by H3$\mathfrak{H}$, $\mathfrak{H} - V_{\alpha(x_i)}$ is closed; consequently if $\mathfrak{H}$ is *regular*, we can so choose $V^*_{\alpha(x_i)}$ that to any $y \subset \mathfrak{H} - V_{\alpha(x_i)}$ corresponds $V_y \subset \mathfrak{H} - V^*_{\alpha(x_i)}$. Defining N^*_α as in HT1 from $x_1, \cdots, x_n$ and $V^*_{\alpha(x1)}, \cdots, V^*_{\alpha(x_i)}$, we see that any $\gamma \subset \bar{G}$ is in some neighborhood N_γ defined by x_i and the requirement $\tau(x_i) \subset V_{\gamma(x_i)} \subset \mathfrak{H} - V^*_{\alpha(x_i)}$, and hence disjoint from N^*_α.

Since finally any sum of open sets is open, this shows that we can include $\bar{G}$ in an open set having no points in common with N^*_α; that is, $\mathfrak{G}$ is regular.

5. **Topological Invariance.** It is important that we establish the topological invariance of $\mathfrak{G}(\mathfrak{H})$ as a function of $\mathfrak{H}$. That is,[8] suppose we substitute for the V_x of §3, W_x having the property that to every x and W_x corresponds a $V_x \subset W_x$, while to every x and V_x corresponds a $W_x \subset V_x$. We want to see whether the M_α defined from the W_x and HT1 have the same property relative to the N_α defined from the V_x and HT1.

This is fairly obvious; for a given $V_{\alpha(x_i)}$ of HT1 we can substitute $W_{\alpha(x_i)} \subset V_{\alpha(x_i)}$; it will follow that the neighborhood M_α so defined will be contained in the N_α defined from the x_i and the $V_{\alpha(x_i)}$. Since the symmetric condition can be proved from analogy, we know

THEOREM 4: *The groupoid* $\mathfrak{G}(\mathfrak{H})$ *of Theorem* 2 *is a topological invariant associated with* $\mathfrak{H}$.

6. **Permutation Groupoids.** Let us now consider the special case that $\mathfrak{H}$ consists of isolated points; that is, that $V_x = x$. In this case HT1 coincides with Kneser's definition[9] of transformation continuity and assumes a special importance. The N are in fact the sets of $(1 \to 1)$ transformations carrying particular finite classes of points of $\mathfrak{H}$ into preassigned transforms. Since $\mathfrak{H}$ is regular, $\mathfrak{G}(\mathfrak{H})$ is a regular Hausdorff space.

By a "Hausdorff" groupoid is meant one in which the elements $\alpha, \beta, \gamma, \cdots$ have "neighborhoods" $N_\alpha, N_\beta, N_\gamma, \cdots$ in the sense of Hausdorff, satisfying H1–H4 and in addition

HG1: To any choice of α, β, and $N_{\alpha\beta}$ correspond N_α and N_β such that $\sigma \subset N$ and $\tau \subset N_\beta$ imply $\sigma\tau \subset N_{\alpha\beta}$.

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[7] E. W. Chittenden, Bull. Am. Math. Soc. 33 (1927), p. 20.

[8] M. Fréchet, "Les espaces abstraits," Paris, 1928, p. 173.

[9] H. Kneser, "Die Deformationssätze der einfach zusammenhängenden Flächen," Math. Zeits. 25 (1926), p. 362.

In the case of permutation groupoids (i.e., the points of $\mathfrak{H}$ are taken as isolated), $\mathfrak{G}(\mathfrak{H})$ satisfies HG1. For, given α, β, and $N_{\alpha\beta}$, we can define N_α and N_β by requiring that their elements transform like α and β respectively, those points $x_1, \cdots, x_n$ which the transformations of $N_{\alpha\beta}$ must transform into $\alpha\beta(x_1), \cdots, \alpha\beta(x_n)$. That is,

THEOREM 5: *The* $(1 \to 1)$ *transformations of any set of isolated points into itself constitute a regular Hausdorff groupoid.*

By a "Hausdorff" group is meant a Hausdorff groupoid which satisfies

G4: Every element α of $\mathfrak{G}$ has an inverse α^{-1} which satisfies $\alpha\alpha^{-1} = \alpha^{-1}\alpha = \underline{1}$.

HG2: To any choice of α and N_α corresponds an $N_{\alpha^{-1}}$ such that $\sigma \subset N_{\alpha^{-1}}$ implies $\sigma^{-1} \subset N_\alpha$.

THEOREM 6:[10] *Any permutation group is a regular Hausdorff group.*

That it is a group is banal; that it is a Hausdorff groupoid we have just seen. It remains to prove that it satisfies HG2. If N_α is defined by requiring that its elements transform $x_1, \cdots, x_n$ like α, then define $N_{\alpha^{-1}}$ by requiring that its elements transform $\alpha(x_1), \cdots, \alpha(x_n)$ like α^{-1}.

7. **Enumerable Case.** Suppose that $\mathfrak{H}$ is enumerable (in symbols, that $o(\mathfrak{H}) = \aleph$); this is a special case of considerable importance. Let us denote the points of $\mathfrak{H}$ by $p_1, p_2, p_3, \cdots$. We can define as follows neighborhoods $U_{n,k}$ of the groupoid $\mathfrak{G}(\mathfrak{H})$.

$U_{n,k}$ $[k = 1, \cdots, n^n; n = 1, 2, 3, \cdots]$ transforms each point p_i $[i = 1, \cdots, n]$ into the point $p_{\phi(i)}$, where $\phi(i)$ denotes the ith digit of k in the n-adic number system. I.e., setting $k = \phi(1)n^{n-1} + \phi(2)n^{n-2} + \cdots + \phi(n)$ $[k = 1, \cdots, n^n]$, then $\tau \subset U_{n,k}$ is equivalent to $\tau(p_i) = p_{\phi(i)}$ $[i = 1, \cdots, n]$.

There are evidently $\aleph$ $U_{n,k}$. Moreover suppose N_α, any neighborhood of $\mathfrak{G}(\mathfrak{H})$; its definition is that $\tau \subset N_\alpha$ means $\tau(p_i) = \alpha(p_i) \equiv p_{\alpha(i)}$ $[i = \theta_1, \cdots, \theta_m]$. Letting n be the largest of the θ_i and $\alpha(i)$, and $k = \sum_{i=1}^{m} \alpha(i)n^{n-\theta_i}$, we have $U_{n,k} \subset N_\alpha$. That is, $\mathfrak{G}(\mathfrak{H})$ is perfectly separable. It follows from results of Tychonoff and Urysohn,[11] that

THEOREM 7: *Let* $\mathfrak{H}$ *consist of* $\aleph$ *isolated points. Then* $\mathfrak{G}(\mathfrak{H})$ *is homeomorphic with a subset of Hilbert space.*

COROLLARY 1: *The Hausdorff group of any group on* $\aleph$ *symbols is homeomorphic with a subset of Hilbert space.*

Moreover, since the $U_{n,k}$ are closed as well as open, $\mathfrak{G}(\mathfrak{H})$ is zero-dimensional. Therefore[12]

COROLLARY 2: $\mathfrak{G}(\mathfrak{H})$ *in Theorem* 7 *is homeomorphic with a line set.*

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[10] Cf. W. Krull, "Galoissche Theorie der unendlichen algebraischen Erweiterungen," Math. Ann. 100 (1928), p. 687.

[11] P. Urysohn, "Zum Metrisationsproblem," Math. Ann. 94 (1925), p. 309. Tychonoff, "Über ein Satz, etc.," Math. Ann. 95 (1925), p. 139.

[12] Cf. K. Menger, "Dimensionstheorie," Berlin, 1929, p. 79 (for dimensions of $\mathfrak{G}(\mathfrak{H})$), and p. 295 (for "Einbettbarkeitssatz").

8. **Meromorphic Product of Enumerable Groups.** Let $\mathfrak{G}$ be the symmetric group on a class Γ containing $\aleph$ symbols. In §§8–9 we shall study some properties of subgroups $\mathfrak{S}$ of $\mathfrak{G}$ which are topologically closed, and whose "transitive systems" are finite.[13]

Let $\Gamma_1, \Gamma_2, \Gamma_3, \cdots$ denote the transitive systems of Γ, S_k $[k = 1, 2, 3, \cdots]$ the normal subgroup of $\mathfrak{S}$ centralizing[14] $\Gamma_1 + \cdots + \Gamma_k$, and s_k the *finite* index of S_{k+1} in S_k.

Let the segment $0 \leqq x \leqq 1$ correspond to the coset of $S_0 = A$ in A. Now use induction, and suppose that to each coset of S_n in A corresponds a segment of length $1/\prod_{k=0}^{n-1} (2s_k - 1)$, while segments corresponding to different cosets are separated. Divide the segment corresponding to any coset of S_n into $2s_n - 1$ equal parts, of which the 1st, 3rd, $\cdots$, $(2s_n - 1)$st are to correspond to the cosets of S_{n+1} in that coset of S_n, while the 2nd, 4th, $\cdots$, $(2s_n - 2)$th are to be empty. This will assign to each coset of S_{n+1} in A a segment of length $1/\prod_{k=0}^{n} (2s_k - 1)$, thus completing the inductive cycle.

The interval-sets corresponding to the cosets of the various S_n $[n = 1, 2, 3, \cdots]$ in $\mathfrak{S}$, converge to a Cantor limit set[15] all of whose points, since $\mathfrak{S}$ is closed, represent elements of $\mathfrak{S}$. Since the representation is continuous as well as (1, 1), we can therefore conclude

THEOREM 8: *Any infinite*[16] *group on* $\aleph$ *symbols whose transitive systems are finite and which is topologically closed, is homeomorphic with a perfect nowhere dense set on the line segment* $0 \leqq x \leqq 1$.

We can moreover define *distance* in $\mathfrak{S}$. We shall set $\overline{\alpha\beta} = 0$ if and only if $\alpha = \beta$; otherwise there will be a first Γ_n on which α and β operate differently, and we shall set $\overline{\alpha\beta} = 1/n > 0$.

Evidently $\overline{\alpha\beta} = \overline{\beta\alpha}$, while $\overline{\alpha\gamma} = \text{Max}\,(\overline{\alpha\beta}, \overline{\beta\gamma}) = \overline{\alpha\beta} + \overline{\beta\gamma}$. $\overline{\alpha\alpha_k} = \overline{\alpha^{-1}\alpha_k^{-1}}$ and $\overline{(\alpha\beta)\,(\alpha_k\beta_k)} \leqq \text{Max}\,(\overline{\alpha\alpha_k}, \overline{\beta\beta_k})$ are also obvious; hence $\alpha_k \to \alpha$ implies $\alpha_k^{-1} \to \alpha^{-1}$, while $\alpha_k \to \alpha$ and $\beta_k \to \beta$ imply $\alpha_k\beta_k \to \alpha\beta$. Finally $\alpha\gamma$ and $\beta\gamma$ permute the symbols of Γ_k alike if and only if α and β do; hence $\overline{(\alpha\gamma)\,(\beta\gamma)} = \overline{\alpha\beta}$, and similarly $\overline{(\gamma\alpha)\,(\gamma\beta)} = \overline{\alpha\beta}$. That is, in summary

THEOREM 9: *Any group on* $\aleph$ *symbols whose transitive systems are finite and which is topologically closed, is an M-group in the sense of van Dantzig.*[17]

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[13] A. Speiser, "Gruppentheorie," 2d. ed., Berlin, 1927, p. 112. What we require is that the number of transforms of any symbol under $\mathfrak{S}$ is finite.

[14] That is, transforming each symbol of $\Gamma_1 + \cdots + \Gamma_k$ into itself. σ of $\mathfrak{S}$ will be said to *normalize* $\Gamma_1 + \cdots + \Gamma_k$ if and only if it transforms $\Gamma_1 + \cdots + \Gamma_k$ into itself.

[15] W. H. Young, "Theory of sets of points," Cambridge, 1906, p. 89.

[16] We have assumed implicitly that we are not dealing with the special case $S_n = 1$ for some n.

[17] D. van Dantzig, "Zur topologische Algebra. I. Komplettierungstheorie," Math. Ann. 107 (1933), p. 608. Cf. for comparison his MG. 10 and MG. 16.

9. **Measure.** Keeping the notation of the last section, let the cosets of S_n be $S_1^n, \cdots, S_{f(n)}^n$; evidently $f(n) = \prod_{k=0}^{n-1} s_k$. Let us define $\mu(S_i^n) = \mu_n \equiv 1/f(n)$. And if Σ: $S_{i(1)}^{n(1)}, S_{i(2)}^{n(2)}, S_{i(3)}^{n(3)}, \cdots$ is any sequence of the S_i^n, by $\mu(\Sigma)$, understand $\sum_{k=1}^{\infty} \mu_{n(k)}$. Evidently $\mu(\Sigma)$ is zero, positive and finite, or $+\infty$.

Relative to any subset H of $\mathfrak{S}$, the sequences Σ fall into two classes—those whose sum contains H (in symbols, $\Sigma \supset H$), and those whose sum does not contain H. Let us *define* that "mass" μ^*H of any H as the lower bound of $\mu(\Sigma)$ for the $\Sigma \supset H$.

Obviously (I) $0 \leqq \mu^*H = 1$, (II) $H' \subset H$ implies $\mu^*H' \leqq \mu^*H$, while (III) if $V = H_1 + H_2 + H_3 + \cdots$, then $\mu^*V \leqq \mu^*H_1 + \mu^*H_2 + \mu^*H_3 + \cdots$. Proofs of these relations can be based on the corresponding proofs for Lebesgue measure.

Moreover suppose there is a positive lower bound $1/N$ to the distance between points of H and points of H', while $\Sigma \supset H + H'$. We can replace the $S_{i(k)}^{n(k)}$ of Σ satisfying $n(k) \leqq N$ by the S_i^N of which they are composed, to form $\Sigma' \supset H + H'$. But $\mu(\Sigma) = \mu(\Sigma')$, and in addition no $S_{i(k)}^{n(k)}$ of Σ' contains points of H *and* points of H'. Separating the $S_{i(k)}^{n(k)}$ of Σ' into those which do and those which do not contain points of H, we therefore have $\mu(\Sigma) = \mu(\Sigma') \geqq \mu^*H + \mu^*H'$. That is, if $\Sigma \supset H + H'$, then $\mu(\Sigma) \geqq \mu^*H + \mu^*H'$; hence $\mu^*(H + H') = \mu^*H + \mu^*H'$. Combining with (III), we see that (IV) if H and H' have a positive separation, then $\mu^*(H + H') = \mu^*H + \mu^*H'$.

Suppose $\Sigma \supset K$, any set of elements of $\mathfrak{S}$. The points in no $S_{i(k)}^{n(k)}$ of Σ for $n(k) \leqq N$ form a *closed* set $(\mathfrak{S} - K)_N$. If $\mu(\Sigma) < 1$, then none of the $(\mathfrak{S} - K)_N$ are empty; hence[18] their meet $(\mathfrak{S} - K)$ is not empty either. Consequently if $\Sigma \supset \mathfrak{S}$, then $\mu(\Sigma) \geqq 1$, showing that $\mu^*\mathfrak{S} \geqq 1$. But setting $\Sigma = S_1^1$, we see that $\mu^*\mathfrak{S} \leqq \mu(\Sigma) = 1$. Therefore $\mu^*\mathfrak{S} = 1$, and by (IV) and symmetry, $\mu^*S_i^n = 1/f(n) = \mu_n$.

The reader can see that[19] irrespective of i, n, and H, $\mu^*H = \mu^*(S_i^n \frown H) + \mu^*((\mathfrak{G} - S_i^n) \frown H)$, whence the S_i^n are measurable in the sense of Caratheodory.[20] Finally, by definition, (V) μ^*H is the lower bound to the measure of the measurable sets of containing H. This proves

THEOREM 10:[21] *$\mathfrak{S}$ and its subsets can be given a regular Caratheodory mass-function, making $\mathfrak{S}$ measurable and of measure* 1.

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[18] Since also $(\mathfrak{S} - K)_{N+1} \subset (\mathfrak{S} - K)_N$; cf. W. H. Young, "The theory of sets of points," p. 89.

[19] By $H \frown K$ is meant the "meet" or common part of H and K; by $H \vee K$, their "join." Cf. E. Steinitz, "Algebraische Theorie der Korper," Berlin, 1930, in the appendix, "Abriss der Galoisschen Theorie," by R. Baer and H. Hasse. This will be cited in later references as [B − H].

[20] Cf. C. Caratheodory, "Vorlesungen über reelle Funktionen," 2d ed., Berlin, 1927, especially pages 238, 258.

[21] Cf. A. Haar, "Der Massbegriff in der Theorie der kontinuierlichen Gruppen," Ann. of Math. 34 (1933), 147–70.

Furthermore let Σ: $S_{i(1)}^{n(1)}$, $S_{i(2)}^{n(2)}$, $S_{i(3)}^{n(3)}$, $\cdots$ be given. By $\Sigma\alpha$ is meant the sequence $S_{i(1)}^{n(1)}\alpha$, $S_{i(2)}^{n(2)}\alpha$, $S_{i(3)}^{n(3)}\alpha$, $\cdots$; by $\alpha\Sigma$, the sequence $\alpha S_{i(1)}^{n(1)}$, etc. Evidently $\mu(\Sigma\alpha) = \mu(\alpha\Sigma) = \mu(\Sigma)$. But given H, $\alpha \subset \mathfrak{S}$ and $\Sigma \supset H$, we know that $\Sigma\alpha \supset H\alpha$ and $\alpha\Sigma \supset \alpha H$. Hence choosing Σ so that $\mu(\Sigma) \leqq \mu^*H + \epsilon$, we know that $\mu^*(\alpha H) \leqq \mu(\alpha\Sigma) = \mu(\Sigma) \leqq \mu^*H + \epsilon$; consequently $\mu^*(\alpha H) = \mu^*H$. But conversely $\mu^*H = \mu^*(\alpha^{-1}\alpha H) \leqq \mu^*(\alpha H)$, whence $\mu^*H = \mu^*(\alpha H)$. Similarly $\mu^*H = \mu^*(H\alpha)$, and

THEOREM 11: *The mass-function of Theorem* 10 *is invariant under group congruences, whether right- or left-handed.*

10. **Continuous Transformations.** A $(1 \to 1)$ transformation α of the Hausdorff space $\mathfrak{H}$ into itself is said to be "continuous" if and only if it satisfies CH1: To any x and $V_{\alpha(x)}$ corresponds a V_x such that $\alpha(V_x) \subset V_{\alpha(x)}$.

THEOREM 12: *The* $(1 \to 1)$ *continuous transformations of any Hausdorff space* $\mathfrak{H}$ *into itself are a subgroupoid of* $\mathfrak{G}(\mathfrak{H})$.

For suppose α and β any two $(1 \to 1)$ continuous transformations of $\mathfrak{H}$ into itself. Given x and $V_{\alpha\beta(x)}$, then (1) we can choose $V_{\alpha(x)}$ so small that $\beta(V_{\alpha(x)}) \subset V_{\alpha\beta(x)}$, and (2) we can choose V_x so small that $\alpha(V_x) \subset V_{\alpha(x)}$. But for this V_x, $\alpha\beta(V_x) \subset \beta(\alpha(V_x)) \subset \beta(V_{\alpha(x)}) \subset V_{\alpha\beta(x)}$. Since the transformation $\underline{1}$ satisfies CH1 (setting $V_x = V_{\alpha(x)}$), by Theorem 1 we see that the $(1 \to 1)$ continuous transformations of $\mathfrak{H}$ into itself are a groupoid, proving the theorem.

A (1,1) transformation of $\mathfrak{H}$ into itself is said to be continuous if and only if it satisfies CH1 and also

CH2: To any x and $V_{\alpha^{-1}(x)}$ corresponds a V_x such that $\alpha^{-1}(V_x) \subset V_{\alpha^{-1}(x)}$.

That is to say, α^{-1} is also $(1 \to 1)$ continuous. Consequently the (1,1) continuous transformations of $\mathfrak{H}$ into itself are a group.

11. **$\mathfrak{K}$-Space.** To express the content of Kneser's[22] general continuity definitions in terms of Hausdorff's neighborhood axioms, we make a new definition.

Let S be any set of $(1 \to 1)$ continuous transformations of the Hausdorff space $\mathfrak{H}$ into itself. The $(1 \to 1)$ continuous transformation α of $\mathfrak{H}$ into itself will be said to be a *K-limit* of S if and only if

HT2: Given x and $V_{\alpha(x)}$ in $\mathfrak{H}$, there exist V_x in $\mathfrak{H}$ and $\alpha \neq \sigma$ in S, satisfying $\sigma(V_x) \subset V_{\alpha(x)}$.

HT2 ascribes to every $\mathfrak{H}$ a "K-space" $\mathfrak{K}(\mathfrak{H})$, whose points correspond (1,1) to those of $\mathfrak{G}(\mathfrak{H})$, but in which the definition of continuity is different.

$\mathfrak{K}(\mathfrak{H})$ has the advantage over $\mathfrak{G}(\mathfrak{H})$ that it satisfies the group property: if α is a K-limit of S, and β of T, then $\alpha\beta$ is a K-limit of ST. That is, given x and $V_{\alpha\beta(x)}$ in $\mathfrak{H}$, we can choose $V_{\alpha(x)}$ and $\tau \subset T$ satisfying $\tau(V_{\alpha(x)}) \subset V_{\alpha\beta(x)}$, and subsequently V_x satisfying $\sigma(V_x) \subset V_{\alpha(x)}$. Hence

$$\sigma\tau(V_x) \equiv \tau(\sigma(V_x)) \subset \tau(V_{\alpha(x)}) \subset V_{\alpha\beta(x)}.$$

[22] *Op. cit.*, Math. Zeits., Vol. 25. This section is to be regarded as descriptive; for instance, the proof that we can make is omitted. Its principal purpose is to relate $\mathfrak{G}(\mathfrak{H})$ to the more important $\mathfrak{K}(\mathfrak{H})$, and to show why we drop the subject.

$\mathfrak{K}(\mathfrak{H})$ is further a *topological invariant* associated with $\mathfrak{H}$, and a very interesting one. But inasmuch as it is not in general a Hausdorff space, it is preferable to discuss it in terms of other definitions. The author proposes to do this in a paper entitled "On the topology of transformation-sets."

12. **Applications.** There are a number of interesting applications of the foregoing theorems. In the first place, we can conclude that every set of (1,1) transformations can be regarded as a regular Hausdorff group: this follows from Theorem 6. Moreover in general the space of the group is not trivial; none[23] of its points is isolated.

We get an even more striking application from Theorems 8–10. Let $\mathfrak{R}$ be any enumerable commutative ring, and let $\mathfrak{A}$ be any ring algebraic over $\mathfrak{R}$. Then since any automorphism of centralizing $\mathfrak{R}$ permutes the roots of the enumerable equations whose coefficients are in $\mathfrak{R}$, while each equation has only a finite number of roots,[24] we can obtain as a direct corollary from these theorems,

THEOREM 13:[25] *Let $\mathfrak{R}$ be any enumerable commutative ring, and let $\mathfrak{A}$ be any ring algebraic over $\mathfrak{R}$. Then the group of $\mathfrak{A}$ over $\mathfrak{R}$ is either finite, or else it is* (i) *a Cantor set on the line segment $0 \leqq x \leqq 1$,* (ii) *an M-group in the sense of van Dantzig, and* (iii) *possessed of a regular mass-function in the sense of Caratheodory.*

13. **Automorphic Extension.** We can weaken the hypotheses of a result of Baer-Hasse,[26] by admitting only fields of the complex number system $\mathfrak{C}$. Specifically, we can affirm

THEOREM 14: *Let $\mathfrak{A}$ be the field of numbers algebraic over $\mathfrak{K}$, let $\mathfrak{F}$ and $\bar{\mathfrak{F}}$ be and fields between $\mathfrak{K}$ and $\mathfrak{A}$, and let ϕ be any* (1,1) *isomorphism normalizing $\mathfrak{K}$ and carrying $\mathfrak{F}$ into $\bar{\mathfrak{F}}$. Then there is an automorphism α of $\mathfrak{A}$ transforming $\mathfrak{F}$ like ϕ.*

Well-ordering $\mathfrak{A}$-$\mathfrak{F}$, we can pass from $\mathfrak{F}$ to $\mathfrak{R}$ by a chain of simple adjunctions. But to each adjunction of the chain corresponds[27] an isomorphic adjunction of $\bar{\mathfrak{F}}$, while we can pass to adjunctions having no preceding adjunction by summation.[28] Therefore we can extend ϕ to α: $\mathfrak{A} \sim \bar{\mathfrak{A}} \subset \mathfrak{A}$. But $\bar{\mathfrak{A}}$ contains as many roots of each $f(x) = 0$ irreducible in $\mathfrak{K}$ as $\mathfrak{A}$ does of $\phi^{-1}(f(x)) = 0$, that is—since degree is invariant under ϕ—all of them. Hence $\bar{\mathfrak{A}} = \mathfrak{A}$, and the theorem is proved.

Suppose now $\mathfrak{S}$ and $\bar{\mathfrak{S}}$ are any two fields of $\mathfrak{C}$, that $\mathfrak{S}$ is algebraic in $\mathfrak{S} \frown \bar{\mathfrak{S}}$, and that there exists a (1,1) isomorphism π: $\mathfrak{S} \sim \bar{\mathfrak{S}}$, carrying $\mathfrak{S} \frown \bar{\mathfrak{S}}$ into itself. Since the field $\mathfrak{A}$ of numbers algebraic in $\mathfrak{G} \frown \bar{\mathfrak{S}}$ is normal and algebraic

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[23] Because of the homogeneity of a Hausdorff group, if one of its points is not isolated, none of its points is isolated.

[24] B. L. van der Waerden, "Moderne Algebra," Berlin, 1930-1, p. 69.

[25] Cf. W. Krull, op. cit., for the most relevant known results.

[26] [$B - H$], Lemma 1a, p. 135. They require that ϕ should centralize $\mathfrak{K}$. Cf. footnote 1, p. 9, and footnote 1, p. 12.

[27] Cf. van der Waerden's "Moderne Algebra," p. 92.

[28] As for instance in [$B - H$], proof of their Lemma 1a.

over $\mathfrak{S} \frown \bar{\mathfrak{S}}$, by Theorem 14 there exists an automorphism α of $\mathfrak{A}$ transforming $\mathfrak{S}$ like π.

We can[29] pass from $\mathfrak{A}$ through pure transcendental extension (of suitable grade t) to a field $\mathfrak{J}$ over which $\mathfrak{C}$ is algebraic. But α can be extended from $\mathfrak{A}$ to $\mathfrak{J}$ since $\mathfrak{J}$ is the field of rational functions in t unknowns whose coefficients are in $\mathfrak{A}$. And by Theorem 14 we can further extend α from $\mathfrak{J}$ to $\mathfrak{C}$, proving

THEOREM 15: *Let π be any $(1,1)$ isomorphism between two fields of $\mathfrak{C}$, normalizing their meet, and suppose that the first field is algebraic over their meet. Then π can be extended into an automorphism of $\mathfrak{C}$.*

Let ω denote an imaginary fifth root of unity, and suppose that $\mathfrak{B}$ is the adjunction of $(\omega + \omega^4)$ and $(\omega^2 + \omega^3)$ to the rational domain $\mathfrak{R}$. There is an automorphism β of $\mathfrak{B}$ of order two on $\mathfrak{B}$—namely, that performed by the cyclic permutation $\gamma\colon \omega \to \omega^2 \to \omega^4 \to \omega^3 \to \omega$. But the only automorphism of $\bar{\mathfrak{B}} = \mathfrak{R} \vee \omega$ which is an extension of β is γ (since the group of $\bar{\mathfrak{B}}$ is cyclic and determined by the j of $\omega \to \omega^j$). Hence

THEOREM 16: *It is not always possible so to extend an automorphism as to preserve order.*

14. **Order of C.** By Theorem 15, the order $o\{C\}$ of the group C of automorphisms of $\mathfrak{C}$ is at least the order of the group A of the automorphisms of the algebraic domain $\mathfrak{A}$, namely,[30] $\mathfrak{c}$. Now let $\mathfrak{B}$ be an irreducible basis for a purely transcendental extension of $\mathfrak{A}/$ into a field $\mathfrak{J}$ over which $\mathfrak{C}$ is algebraic, denote the order $o\{\mathfrak{B}\}$ of $\mathfrak{B}$ by b, and $o\{\mathfrak{J}\}$ by t.

t is at most the number of rational functions in b unknowns with coefficients in $\mathfrak{A}$, that is,[31] $\sum_{k=1}^{\infty}\left(\sum_{i=k}^{\infty} \aleph^{i+k} b^{i+k}\right)$. But since $b \geqq \aleph$, $b\aleph = b$, whence $t = \sum_{k=1}^{\infty}\sum_{i=k}^{\infty} b^{i+k} = \sum_{j=1}^{\infty} j b^j$. Similarly $o\{\mathfrak{C}\} = \mathfrak{c}$ is at most the number of roots of polynomial equations with coefficients in $\mathfrak{J}$, namely, $\sum_{k=1}^{\infty} k t^k$. Combining, we get $\mathfrak{c} \leqq \sum_{k=1}^{\infty} k\left(\sum_{j=1}^{\infty} j b^j\right)^k$ = (by polynomial expansion) $\sum_{i=1}^{\infty} n_i b^i$ where the n_i are finite. Therefore $b > \aleph$. But $o\{\mathfrak{C}\}$ is at least the number of $(1,1)$ transformations of $\mathfrak{B}$ into itself, which is at least 2^b. Hence if we admit the continuum hypothesis, that $b > \aleph$ implies $b \geqq \mathfrak{c}$, then $o\{\mathfrak{C}\} = 2^{\mathfrak{c}}$. But obviously $o\{\mathfrak{C}\} = \mathfrak{c}^{\mathfrak{c}} = (2^{\aleph})^{\mathfrak{c}} = 2^{\aleph\mathfrak{c}} = 2^{\mathfrak{c}}$; consequently by the Theorem of Banach.[32]

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[29] Steinitz, "Algebraische Theorie der Korper," p. 119, 2.

[30] $\mathfrak{c} \equiv$ the power of the continuum; for the fact $o\{A\} = \mathfrak{c}$, see either $[B - H]$, Theorem 3, b), or our own Theorem 13, and Theorems 12 and 13 of Young's "Theory of sets of points," p. 89.

[31] k = leading exponent of denominator, i = that of numerator.

[32] Namely, that $m \geqq n$ and $m \leqq n$ imply $m = n$, even for transfinite numbers.

THEOREM 17: *If we admit the continuum hypothesis, then we admit that the number of automorphisms of the complex number domain is* $2^{\mathfrak{c}}$.

15. **Lattice Theory.** We shall first digress to discuss "Satz 4" of Baer-Hasse,[33] in terms of their own notation and definitions.

Apparently they have overlooked a crucial point in their proof. Their statements concerning their formulae (1) and (3) are valid. But contrary to their assertion, the logic cannot be repeated to yield their (2) or their (4); there is no reason to suppose either (ii) every automorphism centralizing $(\mathfrak{Z}_1 \frown \mathfrak{Z}_2)$ lies in $N(\mathfrak{Z}_1) \vee N(\mathfrak{Z}_2)$, or (iv) every number centralized by $(U_1 \frown U_2)$ lies in $\mathfrak{N}(U_1) \vee \mathfrak{N}(U_2)$. To obtain (2) and (4) from (1) and (3) probably the easiest way is to restrict ourselves to subgroups and subfields for which their (F) of Hauptsatz 3 holds. For these we can prove, setting $U_i = N(\mathfrak{Z}_i)$ and $\mathfrak{Z}_i = \mathfrak{N}(U_i)$,

$$N(\mathfrak{Z}_1 \frown \mathfrak{Z}_2) = N(\mathfrak{N}(U_1) \frown \mathfrak{N}(U_2)) = N(\mathfrak{N}(U_1 \vee U_2)) = U_1 \vee U_2 = N(\mathfrak{Z}_1) \vee N(\mathfrak{Z}_2)$$

and

$$\mathfrak{N}(U_1 \frown U_2) = \mathfrak{N}(N(\mathfrak{Z}_1) \frown N(\mathfrak{Z}_2)) = \mathfrak{N}(N(\mathfrak{Z}_1 \vee \mathfrak{Z}_2)) = \mathfrak{Z}_1 \vee \mathfrak{Z}_2 = \mathfrak{N}(U_1) \vee \mathfrak{N}(U_2).$$

Since, as Baer-Hasse show, (F) holds for subgroups of A of finite index and the corresponding finite adjunctions onto the rational domain $\mathfrak{R}$, we can reëxpress their formulae, as

THEOREM 18: *The lattice*[34] $\overline{\mathfrak{L}}$ *of the subgroups of* A *of finite index in* A, *and the lattice* $\mathfrak{L}$ *of fields of finite grade over* $\mathfrak{R}$, *are "symmetric counterparts."*[35]

And because of the symmetry in the definition of B-lattices, and that between normal subgroups of A and normal extensions of $\mathfrak{R}$, we can conclude[36]

THEOREM 19: *The normal fields of finite grade over* $\mathfrak{R}$ *are a* B*-lattice.*

COROLLARY:[37] *Given any three normal algebraic extensions* $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{N}_3$ *of finite grade over the rational domain, the number of fields definable from the* $\mathfrak{N}_i$ *by intersection, adjunction with each other, and induction, is at most twenty-eight.*

Finally, the adjunctions of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$ to the rational domain do not obey the distributive law, showing that the B-lattice of Theorem 19 is not a C-lattice.

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[33] p. 143 of Steinitz's "Algebraische Theorie der Korper;" the following paragraph is of import and intelligible only when it is compared with the text there.

[34] For the theory of "lattices," cf. the author's "On the combination of subalgebras," Proc. Camb. Phil. Soc., 29, pp. 441–464. Here A is understood in the sense of the last section.

[35] "Subalgebras," Theorem 4.1.

[36] $[B - H]$, Satz 1 and Zusatz 1, and "Subalgebras," Theorem 26.1.

[37] "Subalgebras," Theorem 14.1. Other corollaries could be given based upon, for instance, Theorem 26.4.

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THE THEORY OF TOPOLOGICAL COMMUTATIVE GROUPS

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Introduction

The purpose of the present paper is to make an exhaustive investigation into the structure of continuous, locally compact, commutative groups, satisfying the second axiom of countability.[2]

1. **Contents.** First of all I discuss compact groups. The principal method here is the determination of the correspondence between continuous and discrete groups: To every discrete commutative group $\mathfrak{G}$ corresponds a compact commutative group X, which is called the group of characters of the group $\mathfrak{G}$. The first chapter of this paper is devoted to the study of the connection between the groups $\mathfrak{G}$ and X. It is also written so as to be applicable to combinatorial topology. In the second chapter I show that every compact, commutative group is the group of characters of some discrete group, using results of a paper by F. Peter and H. Weyl,[3] and also results of Haar.[4] The one-to-one correspondence thus established between compact commutative groups and discrete commutative groups enables us to reduce the study of the former to that of the latter. This fact is illustrated by the first and second appendices to this paper.

In the first appendix I establish a complete equivalence between the question of decomposition of compact, commutative groups into a direct sum, and the corresponding question concerning discrete commutative groups. I also give here some counter examples. The second appendix gives the connection between the simplest topological properties of the group X and the algebraic properties of the group $\mathfrak{G}$.

In Chapter 3 I consider locally compact groups (which need not be compact) and confine myself to the case of connected groups. I show, first of all, that every such group Ω admits a subgroup Δ which has no limiting elements in Ω, and which is such that the factor group Ω/Δ is compact. This result, which is proved in a purely topological way, enables us to reduce the study of non-compact groups to the consideration of compact groups, which have already

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[1] The results of this paper were reported in condensed form at the International Mathematical Congress in Zurich, 1932.

[2] For the fundamental concepts of the theory of continuous i.e. topological groups see D. van Dantzig, *Zur topologischen Algebra* 1, Math. Ann. 107 (1932), pp. 587–626.

[3] F. Peter and H. Weyl: *Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe.* Math. Ann. 97 (1927) pp. 737.

[4] A. Haar: *Der Massbegriff in der Theorie der kontinuierlichen Gruppen.* These Annals (2) 34 (1933) pp. 147–169.

been discussed. In the same way I show that the group Ω is decomposible into the sum of a certain compact group and a system of groups isomorphic to the additive group of real numbers.

In the fourth and last section I investigate the structure of locally connected, locally compact, connected groups. It turns out that every such group is the direct sum of a finite or denumerable number of groups K and a finite number of groups isomorphic to the additive group of real numbers. The group K is the so-called continuous cyclic group, that is an additive group of real numbers determined up to an additive integer.

2. **Notation.** A continuous cyclic group, namely the additive group of real numbers determined up to an additive integer, is of fundamental importance in this paper, since with its aid we obtain the notion of the group of characters. We designate this group by K.

All the groups considered in this paper are commutative and are written additively except in Chapter 2. All the continuous groups discussed are locally compact and satisfy the second axiom of countability. All the discrete groups contain at most a denumerable number of elements.

Continuous groups are designated by capital Greek letters and their elements by small Greek letters. Discrete groups are designated by capital German letters, and their elements by small German letters.

I. Group of Characters

Here I discuss the connection between discrete and continuous groups and show that to each discrete group there corresponds a certain continuous group and conversely; the correspondence being reciprocal. Moreover, this pairing of the groups enables me to establish a correspondence between their sub-groups.

All continuous groups considered here are compact.

1. Definition 1. Let K be the continuous group of rotations of a circle, which we will consider as an additive group of real numbers defined up to an additive integer, and let $\mathfrak{G}$ be a discrete group.

Let us denote by X the set of all homomorphic representations of the group $\mathfrak{G}$ into the group K. The set X forms a continuous group in the following natural way. A sequence

$$\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$$

of elements of X is said to converge to the element α, if for each $\mathfrak{x} \subset \mathfrak{G}$,

$$\lim_{n\to\infty} \alpha_n(\mathfrak{x}) = \alpha(\mathfrak{x}).^5$$

We call the sum of two elements α' and α'' an element α such that

$$\alpha'(\mathfrak{x}) + \alpha''(\mathfrak{x}) = \alpha(\mathfrak{x}),$$

[5] By $\alpha(\mathfrak{x})$ I understand the function which puts into correspondence any element $\mathfrak{x} \subset \mathfrak{G}$ with the element $\alpha(\mathfrak{x})$ of the group K, which is determined by the homomorphism.

for each $\mathfrak{x} \subset \mathfrak{G}$. It is easy to see that the operations thus defined satisfy all the axioms of a continuous group. The group X so obtained we shall call the *group of characters* of the group $\mathfrak{G}$.

REMARK 1. If $\mathfrak{G}$ is a finite group the definition of a group of characters given here coincides with the usual one. In fact, usually the group of characters of $\mathfrak{G}$ is defined as the group of homomorphisms of $\mathfrak{G}$ on the multiplicative group of complex numbers; but as all the elements of $\mathfrak{G}$, by virtue of the finiteness of $\mathfrak{G}$, have a finite order, they must be mapped onto complex numbers of absolute value 1. But the multiplicative group of complex numbers of absolute value 1 is isomorphic to the group K: In fact, let z be a complex number $|z| = 1$. Let us bring it into correspondence with a real number, equal to $\log z/2\pi i$. This number is defined up to an additive integer and constitutes an element of the group K; it is easy to see that the correspondence obtained in this way is an isomorphism.

THEOREM 1. *The group of characters of a discrete group is always compact and satisfies the second axiom of countability.*

PROOF. Let

$$\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$$

be an arbitrary sequence of elements of the group X and

$$\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n, \cdots$$

be the ordered sequence of all elements of the group $\mathfrak{G}$. Consider the sequence

$$\alpha_1(\mathfrak{x}_1), \alpha_2(\mathfrak{x}_1), \cdots, \alpha_n(\mathfrak{x}_1), \cdots$$

of elements of the group K. Since K is compact there exists a sub-sequence

$$\alpha_{i_1}(\mathfrak{x}_1), \alpha_{i_2}(\mathfrak{x}_1), \cdots, \alpha_{i_n}(\mathfrak{x}_1), \cdots$$

converging to a limit element in K. Let us now consider the sequence

$$\alpha_{i_1}(\mathfrak{x}_2), \alpha_{i_2}(\mathfrak{x}_2), \cdots, \alpha_{i_n}(\mathfrak{x}_2), \cdots$$

from it we can again choose a convergent sub-sequence

$$\alpha_{i_{j_1}}(\mathfrak{x}_2), \alpha_{i_{j_2}}(\mathfrak{x}_2), \cdots, \alpha_{i_{j_n}}(\mathfrak{x}_2), \cdots.$$

Continuing in this way and applying, finally the diagonal process, that is, constructing the sequence

$$\alpha_1, \alpha_{i_2}, \alpha_{i_{j_3}}, \cdots$$

we arrive at a converging sequence of elements of X, thereby proving the compactness of X.

To prove the second axiom of countability let us give the topological structure of X by means of neighborhoods. This can be done in the following way: Let $\alpha \subset X$ and let us define a certain neighborhood U of the element α as follows:

let I_s be a certain interval on K containing $\alpha(\mathfrak{x}_s)$, $(s = 1, 2, \cdots, m)$, such that the ends of the interval are rational numbers. Then, by definition, $\beta \subset U$, if

$$\beta(\mathfrak{x}_s) \subset I_s.$$

It is easy to see that the definition of neighborhoods given here leads to the same topological structure of X as that given in Definition 1, but the axiom of countability is here evidently fulfilled, since only a countable system of neighborhoods of X is given.

2. LEMMA 1. *Let $\mathfrak{G}$ be a group, $\mathfrak{H}$ a sub-group of $\mathfrak{G}$, $\mathfrak{x}$ a certain element of $\mathfrak{G}$, $\not\subset \mathfrak{H}$, and $\alpha \neq 0$ a certain homomorphism of $\mathfrak{H}$ in* K (*as always,* K *is the same as in Definition* 1). *The homomorphism α can be extended to a homomorphism β of the whole group $\mathfrak{G}$ in* K, $\beta(\mathfrak{x}) \neq 0$.

PROOF. Let

$$\mathfrak{x}_1 = \mathfrak{x}, \mathfrak{x}_2, \mathfrak{x}_3, \cdots, \mathfrak{x}_n, \cdots$$

be the totality of all elements of the group $\mathfrak{G}$. Let us construct the sequence

$$\mathfrak{H}_0 = \mathfrak{H}, \mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_n,$$

of sub-groups of $\mathfrak{G}$, where $\mathfrak{H}_{n+1}$ is the minimal sub-group containing $\mathfrak{H}_n$ and $\mathfrak{x}_{n+1}$. We may then construct further the sequence of homomorphisms

$$\alpha_0 = \alpha, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$$

where α_{n+1} is a homomorphism of the group $\mathfrak{H}_{n+1}$ in K, which is an extension of the homomorphism of $\mathfrak{H}_n$. Suppose that α_n is already constructed, and let us construct α_{n+1}. If $\mathfrak{x}_{n+1} \subset \mathfrak{H}_n$ then we set $\alpha_{n+1} = \alpha_n$, since in this case $\mathfrak{H}_{n+1} = \mathfrak{H}_n$. If $\mathfrak{x}_{n+1} \not\subset \mathfrak{H}_n$ then two cases arise: 1) $\mathfrak{x}_{n+1}$ is linearly independent of the elements of $\mathfrak{H}_n$, that is, there do not exist non-trivial relations of the form $\mathfrak{h} + k\mathfrak{x}_{n+1} = 0$, where $\mathfrak{h} \subset \mathfrak{H}_n$, and k is an integer. 2) $\mathfrak{x}_{n+1}$ depends linearly on the elements of the group $\mathfrak{H}_n$. In the first case each element of $\mathfrak{H}_{n+1}$ is uniquely expressed in the form $\mathfrak{h} + k\mathfrak{x}_{n+1}$. Then we put

$$\alpha_{n+1}(\mathfrak{h} + k\mathfrak{x}_{n+1}) = \alpha_n(\mathfrak{h}) + k\gamma$$

where γ is an arbitrary element of K (when $n = 0$ we choose $\gamma \neq 0$). In the second case there are non-trivial relations of the form $\mathfrak{h} + k\mathfrak{x}_{n+1} = 0$. We will choose that relation for which k has the least positive value l, and let it be $\mathfrak{h} + l\mathfrak{x}_{n+1} = 0$, where $l > 1$, for otherwise $\mathfrak{x}_{n+1} \subset \mathfrak{H}_n$. In this case each element of $\mathfrak{H}_{n+1}$ is uniquely expressed in the form $\mathfrak{h} + k\mathfrak{x}_{n+1}$, $0 \leqq k < l$. Let

$$\gamma = \alpha_n(\mathfrak{h})/l$$

(in the group K division is always possible, though perhaps not unique. In fact each element of K is a real number, defined up to an additive integer; therefore, if we divide by an element not unity, it can always be supposed that the quo-

tient is not equal to 0, even when the dividend is 0). (When $n = 0$ we choose $\gamma \neq 0$.) Let us put

$$\alpha_{n+1}(\mathfrak{b} + k\mathfrak{x}_{n+1}) = \alpha_n(\mathfrak{b}) + k\gamma.$$

In this way α_{n+1} is constructed. The sequence of homomorphisms

$$\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$$

defines a homomorphism β of the group $\mathfrak{G}$ in K which does not transform $\mathfrak{x}_1$ into 0.

REMARK 2. Lemma 1 is applicable also in the case when $\mathfrak{G}$ has more than a countable number of elements, except that, instead of the usual induction, it is necessary to apply transfinite induction in the proof.

DEFINITION 2. Let $\mathfrak{G}$ be a group and X be the group of its characters, $\mathfrak{H}$ a sub-group of $\mathfrak{G}$, and Φ a sub-group of X (as was stated, we shall consider only compact continous groups, consequently Φ is compact and closed in X). The totality of all elements of X mapping $\mathfrak{H}$ on 0, we will designate by $(X, \mathfrak{H})$, the totality of all elements of $\mathfrak{G}$ that are mapped on 0 by all homomorphisms of Φ we will designate by $(\mathfrak{G}, \Phi)$. It is easy to see, that $(X, \mathfrak{H})$ and $(\mathfrak{G}, \Phi)$ are groups, and that the first of them is closed in X.

THEOREM 2. *Let $\mathfrak{G}$ be a group, X the group of its characters and $\mathfrak{H}$ be a sub-group of the group $\mathfrak{G}$. If $\Phi = (X, \mathfrak{H})$, then $\mathfrak{H} = (\mathfrak{G}, \Phi)$.*

PROOF. Let $\bar{\mathfrak{H}} = (\mathfrak{G}, \Phi)$. It is obvious that $\mathfrak{H} \subset \bar{\mathfrak{H}}$. Suppose that there exists an element $\mathfrak{x} \subset \bar{\mathfrak{H}}$, such that $\mathfrak{x} \not\subset \mathfrak{H}$. Let α be a homomorphism of group $\mathfrak{H}$ in K mapping all the elements of $\mathfrak{H}$ into 0. According to Lemma 1, there exists a homomorphism β of group $\mathfrak{G}$ in K, which is an extension of α and such that $\beta(\mathfrak{x}) \neq 0$. Since β is an extension of α it transforms $\mathfrak{H}$ into 0, that is, $\beta \subset \Phi$. But, by Definition 2 $\beta(\mathfrak{x}) = 0$, since $\mathfrak{x} \subset \bar{\mathfrak{H}} = (\mathfrak{G}, \Phi)$. Hence, we arrived at a contradiction.

THEOREM 3. *Let $\mathfrak{G}$ be a group, $\mathfrak{H}$ a sub-group of $\mathfrak{G}$, and X the group of characters of $\mathfrak{G}$. Let us put $\Phi = (X, \mathfrak{H})$. Then the group of characters of $\mathfrak{H}$ is isomorphic to the factor group X/Φ, and the group of characters of the factor group $\mathfrak{G}/\mathfrak{H}$ is isomorphic to the group Φ.*

PROOF. Each homomorphism of the group $\mathfrak{G}$ in K gives a certain homomorphism of the group $\mathfrak{H}$ in K, while, according to Lemma 1, every homomorphism of the group $\mathfrak{H}$ may be obtained in this way. On the other hand, two homomorphisms α and β of the group $\mathfrak{G}$ coincide in $\mathfrak{H}$ if and only if their difference $\alpha - \beta$ maps $\mathfrak{H}$ on 0, that is $\alpha - \beta \subset \Phi$. In this way we have a homomorphism of the group X on the group of characters of the group $\mathfrak{H}$, while the totality of all elements of X that are mapped on 0 by this homomorphism, coincides with Φ. Hence, the first part of the theorem is proved.

Each homomorphism of Φ, that is, one by which $\mathfrak{H}$ is mapped on 0, gives a certain homomorphism of the group $\mathfrak{G}/\mathfrak{H}$ in K, but every homomorphism of the group $\mathfrak{G}/\mathfrak{H}$ may be obtained in this way, consequently the second part of the theorem is proved.

3. LEMMA 2. *Each sub-group Ψ of the group* K *either coincides with* K *or is a finite cyclic group with the numerical generator* $1/r$, *where* r *is the order of the group* Ψ. (As usual, we consider only compact groups; that is, Ψ is a closed sub-group of K.)

PROOF. If Ψ contains an infinity of elements, then by virtue of the compactness of K there are two arbitrarily near elements α and β in Ψ. The element $\gamma = \beta - \alpha \subset \Psi$ is arbitrarily small and its integer multiples fill K arbitrarily densely. Thus Ψ is everywhere dense in K and, as Ψ is closed, $\Psi = K$.

If Ψ is of a finite order r, $r\alpha = 0$ for all $\alpha \subset \Psi$. Consequently, each element of Ψ has the form q/r, where q is an integer, but as there are only r such elements in K, for $q = 0, 1, \cdots, r-1$, and they form a group with the generator $1/r$, the lemma is proved.

LEMMA 3. *If the group* $\mathfrak{G}$ *is finite and of order* r, *the group of its characters is also of order* r.

REMARK 3. In reality the group of characters of the finite group $\mathfrak{G}$ is isomorphic to $\mathfrak{G}$. This is a well-known fact, but we will make no use of it.

PROOF. Express the group $\mathfrak{G}$ in the form of a direct sum of cyclic groups $\mathfrak{G}_1, \mathfrak{G}_2, \cdots, \mathfrak{G}_n$, where $\mathfrak{G}_i$ has a generator $\mathfrak{g}_i$ of order r_i. In the mapping on K, the generator of $\mathfrak{G}_i$ is transformed into an element whose order is a divisor of r_i, but as the number of such elements in K, according to Lemma 2 is exactly r_i, the number of possible representations of the generator of $\mathfrak{G}_i$ is equal to r_i, and consequently, the number of possible mappings of $\mathfrak{G}$ in K is equal to the product $r_1 r_2 \cdots r_n$ which is equal to r.

LEMMA 4. *Let* $\mathfrak{G}$ *be a group,* X *the group of its characters, and* Φ *a sub-group of* X, *such that* $(\mathfrak{G}, \Phi) = (0)$, *then* $\Phi = X$.

To prove this lemma we must prove first three preliminary lemmas.

LEMMA 5. *The assertion of Lemma* 4 *holds when* $\mathfrak{G}$ *is finite.*

PROOF. Each element $\mathfrak{x}$ of the group $\mathfrak{G}$ gives a homomorphism of the group Φ in K. In fact, when α runs over Φ, $\alpha(\mathfrak{x})$ is a homomorphism of Φ in K. Since $(\mathfrak{G}, \Phi)$ equals (0), all these homomophisms are distinct, and if r designates the order of $\mathfrak{G}$, their number is r. If the group Φ does not coincide with X, its order $s < r$, so that according to Lemma 3, the group Φ admits only s different homomorphisms in K, of which there exist r, so that $s = r$, a contradiction.

LEMMA 6. *Lemma* 4 *holds when* $\mathfrak{G}$ *admits a finite system of linearly independent generators.*

PROOF. Let $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n$ be a system of linearly independent generators of $\mathfrak{G}$. Let us show that for every k, $0 \leqq k \leqq n$, no matter what elements δ_i, $(i = 1, 2, \cdots, k)$ we choose in K there exists an element $\gamma \subset \Phi$ such that $\gamma(\mathfrak{x}_i) = \delta_i$. When $k = n$ this assertion is equivalent to the assertion of the lemma, for, in this case, it follows that Φ contains all the homomorphisms of $\mathfrak{G}$ in K.

When $k = 0$ our statement is trivial; let us suppose that it holds for a given k and let us prove it for $k + 1$.

Consider the totality Ψ of elements of K expressible as $\alpha(\mathfrak{x}_{k+1})$, $\alpha \subset \Phi$, with the condition that $\alpha(\mathfrak{x}_i) = 0$, $(i = 1, 2, \cdots, k)$.

It is obvious that Ψ is a group. By Lemma 2, Ψ either coincides with K or all of its elements are expressible in the form q/r, where q is an integer and r is the order of Ψ.

If $\Psi = \mathrm{K}$, then the supposition is proved. In fact, let δ_j, $(j = 1, 2, \cdots, k+1)$ be an arbitrary system of elements of K. By the supposition there exists an element $\beta \subset \Phi$, such that $\beta(\mathfrak{x}_i) = \delta_i$, $(i = 1, 2, \cdots, k)$. Suppose, that $\beta(\mathfrak{x}_{k+1}) = \delta'_{k+1}$. Since $\Psi = \mathrm{K}$ there exists an element $\alpha \subset \Phi$, such that $\alpha(\mathfrak{x}_i) = 0$, $\alpha(\mathfrak{x}_{k+1}) = \delta_{k+1} - \delta'_{k+1}$. For $\gamma = \alpha + \beta$ we have $\gamma(\mathfrak{x}_j) = \delta_j$, $(j = 1, 2, \cdots, k+1)$.

Let us now suppose that all the elements of Ψ are expressible in the form q/r, with fixed r. From this it follows, that if $\alpha(\mathfrak{x}_i) = \beta(\mathfrak{x}_i)$, $\alpha \subset \Phi$, $\beta \subset \Phi$, $(i = 1, 2, \cdots, k)$, then

$$\beta(\mathfrak{x}_{k+1}) - \alpha(\mathfrak{x}_{k+1}) = q/r.$$

Let ϵ_i, $(i = 1, 2, \cdots, k)$, be a system of linearly independent elements of K. Let $\gamma(\mathfrak{x}_i) = \epsilon_i$, $\gamma \subset \Phi$, $(i = 1, 2, \cdots, k)$, which is possible according to the induction. Put $\gamma(\mathfrak{x}_{k+1}) = \epsilon_{k+1}$. Let us show that ϵ_j, $(j = 1, 2, \cdots, k+1)$ are linearly dependent. If it were otherwise, then, by the Kronecker theorem, no matter what the system of elements δ_j of K, there could be found an infinite sequence of integers n_m, $(m = 1, 2, \cdots)$, such that

$$\lim_{m \to \infty} n_m \epsilon_j = \delta_j.$$

If we denote by λ one of the limit elements of the sequence $n_m \gamma$, we will have $\lambda(\mathfrak{x}_j) = \delta_j$ where $\lambda \subset \Phi$, since Φ is closed, that is, the induction is proved for $k+1$, and Ψ coincides with K. In this way we have the relation

$$\sum_{j=1}^{k+1} r_j \epsilon_j = 0, \qquad r_j \text{ integers.}$$

Let us show that from this relation there follows

$$\sum_{j=1}^{k+1} r\, r_j\, \alpha(\mathfrak{x}_j) = 0$$

for all $\alpha \subset \Phi$. In fact, since ϵ_i are linearly independent, by Kronecker's theorem there exists an infinite sequence $\bar{n}_m$ of integers, such that $\lim_{m \to \infty} \bar{n}_m \epsilon_i = \alpha(\mathfrak{x}_i)$. The sequence $\bar{n}_m$ can be chosen in such a way that $\bar{n}_m \gamma$ has a limit. Let it be equal to β. It is obvious, then, that

$$\sum_{j=1}^{k+1} r_j \beta(\mathfrak{x}_j) = 0.$$

Since $\alpha(\mathfrak{x}_i) = \beta(\mathfrak{x}_i)$, then, according to what was previously proved

$$\beta(\mathfrak{x}_{k+1}) - \alpha(\mathfrak{x}_{k+1}) = q/r;$$

thus, we have for α:

$$\sum_{j=1}^{k+1} r\, r_j\, \alpha(\mathfrak{x}_j) = 0.$$

Consequently,

$$\alpha\left(\sum_{j=1}^{k+1} r\, r_j\, \mathfrak{x}_j\right) = 0;$$

but, since α is an arbitrary element of Φ therefore

$$(\mathfrak{G}, \Phi) \supset \sum_{j=1}^{k+1} r\, r_j\, \mathfrak{x}_j = \mathfrak{x}.$$

But then, according to the condition of the lemma, $\mathfrak{x} = 0$, which is impossible since the generators $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n$ are linearly independent.

LEMMA 7. *Lemma 4 holds for any $\mathfrak{G}$ with a finite system of generators.*

PROOF. Let $\mathfrak{H}$ be the group composed of all elements of finite order of the group $\mathfrak{G}$. Let us put $\Psi = (X, \mathfrak{H})$, and let us show that $\Psi \subset \Phi$. To this effect let us denote by Δ the intersection of Φ and Ψ and let us show that $\Delta = \Psi$. Let r be the order of $\mathfrak{H}$; let us show that if $\alpha \subset X$, then $r\alpha \subset \Psi$. In fact, for any $\mathfrak{x} \subset \mathfrak{H}$ we have $r\alpha(\mathfrak{x}) = \alpha(r\mathfrak{x}) = \alpha(0) = 0$. Let us show, further, that $(\mathfrak{G}, \Delta) = \mathfrak{H}$. In fact, if we suppose the contrary, there exists a free element $\mathfrak{y}$ of $\mathfrak{G}$ such that for any $\beta \subset \Delta$, $\beta(\mathfrak{y}) = 0$. From this supposition it follows that for any $\alpha \subset \Phi$, $\alpha(r\mathfrak{y}) = r\alpha(\mathfrak{y}) = 0$. $r\alpha \subset \Delta$, because it is contained, as was previously proved, in Ψ and is a member of Φ and so belongs to their intersection. Thus, $(\mathfrak{G}, \Phi) \supset r\mathfrak{y} \neq 0$, which is impossible. Thus, $(\mathfrak{G}, \Delta) = \mathfrak{H}$. By Theorem 3, Ψ is the group of characters of the group $\mathfrak{G}/\mathfrak{H}$. From $(\mathfrak{G}, \Delta) = \mathfrak{H}$ follows $(\mathfrak{G}/\mathfrak{H}, \Delta) = (0)$. Since $\mathfrak{G}/\mathfrak{H}$ is a free group with a finite number of generators it follows, by Lemma 6, that $\Delta = \Psi$, that is, $\Psi \subset \Phi$. Let us set $X/\Psi = \bar{X}$, $\Phi/\Psi = \bar{\Phi}$. Then let us prove that $\bar{\Phi} = \bar{X}$. By Theorem 3, $\bar{X}$ is the group of characters of the group $\mathfrak{H}$. From $(\mathfrak{G}, \Phi) = 0$ it follows that $(\mathfrak{H}, \bar{\Phi}) = 0$, but, since $\mathfrak{H}$ is a finite group, we may apply Lemma 5, from which it follows that $\bar{\Phi} = \bar{X}$.

From $\Psi \subset \Phi$ and from $\Phi/\Psi = X/\Psi$ follows $\Phi = X$.

PROOF OF LEMMA 4. Let $\mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_n, \cdots$ be an expanding sequence of sub-groups of $\mathfrak{G}$, where each $\mathfrak{H}_n$ has a finite number of generators while the sum of all the $\mathfrak{H}_n$ coincides with $\mathfrak{G}$; such a sequence always exists since $\mathfrak{G}$ has not more than a countable number of elements. Let us put $\Psi_n = (X, \mathfrak{H}_n)$. It is obvious that $\Psi_{n+1} \subset \Psi_n$. From the fact that the sum of all $\mathfrak{H}_n = \mathfrak{G}$ it follows that the intersection of all the Ψ_n contains only zero.

Let Φ_n be the group composed of all elements of the form $\alpha + \beta$, where $\alpha \subset \Psi_n$ and $\beta \subset \Phi$. By Theorem 3, X/Ψ_n is the group of characters of the group $\mathfrak{H}_n$. From $(\mathfrak{G}, \Phi) = (0)$ it follows that $(\mathfrak{H}_n, \Phi_n/\Psi_n) = (0)$, but since $\mathfrak{H}_n$

has a finite system of generators it follows, from what has been said, in accordance with Lemma 7, that $\Phi_n/\Psi_n = X/\Psi_n$, that is $\Phi_n = X$. Thus, the intersection of all Φ_n coincides with X, but the same intersection must coincide with Φ, since the intersection of all Ψ_n contains only zero, that is $\Phi = X$.

THEOREM 4. *If $\mathfrak{G}$ is a group, X the group of its characters, Φ a sub-group of* X, $\mathfrak{H} = (\mathfrak{G}, \Phi)$, *then* $(X, \mathfrak{H}) = \Phi$.

PROOF. Let $\bar{\Phi} = (X, \mathfrak{H})$. It is evident that $\bar{\Phi} \supset \Phi$. By Theorem 3, $\bar{\Phi}$ is the group of characters of the group $\mathfrak{G}/\mathfrak{H}$. From $(\mathfrak{G}, \Phi) = \mathfrak{H}$ it follows that $(\mathfrak{G}/\mathfrak{H}, \Phi) = (0)$; from this by Lemma 7 we have $\Phi = \bar{\Phi}$.

4. DEFINITION 1'.[6] Let X be a continuous compact group. The set $\mathfrak{G}$ of its homomorphic mappings in K, forms, in a natural way, just as in Definition 1, a group which we will call the group of characters of group X. We introduce only algebraic operations in $\mathfrak{G}$ and do not consider topological operations; but the mapping of X in K must be continuous.

DEFINITION 3. Let X and $\mathfrak{G}$ be two groups for which there is defined a law of multiplication, that is, to each pair of elements $\alpha \subset X$, $\mathfrak{x} \subset \mathfrak{G}$ corresponds as their product the element $\gamma \subset K$, $\alpha\mathfrak{x} = \gamma$. Furthermore,

$$(\alpha' - \alpha'')\,\mathfrak{x} = \alpha'\mathfrak{x} - \alpha''\mathfrak{x}, \qquad \alpha(\mathfrak{x}' - \mathfrak{x}'') = \alpha\mathfrak{x}' - \alpha\mathfrak{x}'',$$

and if $\lim \alpha_n = \alpha$ then $\lim \alpha_n \mathfrak{x} = \alpha\mathfrak{x}$. Under these conditions we shall say, that the groups X and $\mathfrak{G}$ form a couple. Further we shall introduce notations, analogous to those given in Definition 2; namely: If $\mathfrak{H}$ is a sub-group of $\mathfrak{G}$ and Φ a sub-group of X we shall designate by $(X, \mathfrak{H})$ the set of all elements of X, the products of which with each element of $\mathfrak{H}$ are equal to zero, in the same way $(\mathfrak{G}, \Phi)$ shall designate the set of all elements of $\mathfrak{G}$ the products of which with any element of Φ are equal to zero. In case X and $\mathfrak{G}$ form a couple, such that $(X, \mathfrak{G}) = (0)$, and $(\mathfrak{G}, X) = (0)$ we shall call X and $\mathfrak{G}$ orthogonal.

REMARK 4. It is easy to see, that if X is the group of characters of the group $\mathfrak{G}$, then X and $\mathfrak{G}$ form an orthogonal couple; it is sufficient to put $\alpha\mathfrak{x} = \alpha(\mathfrak{x})$, $\alpha \subset X$, $\mathfrak{x} \subset \mathfrak{G}$ and we obtain the law of multiplication, satisfying all the conditions of Definition 3.

THEOREM 5. *If* X *and $\mathfrak{G}$ are orthogonal, each of these groups is the group of characters of the other; in fact, if $\alpha \subset X$, $\mathfrak{x} \subset \mathfrak{G}$, the homomorphism $\alpha(\mathfrak{x})$ is defined by the equation $\alpha(\mathfrak{x}) = \alpha\mathfrak{x}$ and the homomorphism $\mathfrak{x}(\alpha)$ by the condition $\mathfrak{x}(\alpha) = \alpha\mathfrak{x}$.*

PROOF. Each element $\alpha \subset X$ gives a homomorphism of $\mathfrak{G}$ in K. Moreover to distinct elements of X correspond distinct homomorphisms, since $(X, \mathfrak{G}) = 0$. Thus X is a sub-group of the group $\bar{X}$ of characters of the group $\mathfrak{G}$. On account of the orthogonality of X and $\mathfrak{G}$ it follows that $(\mathfrak{G}, X) = (0)$, and from this by Lemma 4 follows that $X = \bar{X}$.

Similiarly let us prove that $\mathfrak{G}$ is a group of characters for X; here, however, the proof becomes somewhat more complicated, since, *a priori*, the group of

[6] This definition was given in my paper (Note 1).

characters for X could turn out to be uncountable. Therefore, let us carry out the proof somewhat differently. Let $\mathfrak{g}$ be a certain homomorphism of X in K. Let us show that it is realized with the help of one of the elements of group $\mathfrak{G}$. To this effect let us construct a group $\bar{\mathfrak{G}}$, each element of which is expressible in the form $\mathfrak{x} + k\mathfrak{g}$, where $\mathfrak{x} \subset \mathfrak{G}$, and k is an integer; let us define the law of multiplication of an arbitrary element $\alpha \subset X$ by an arbitrary element $\mathfrak{x} + k\mathfrak{g} \subset \bar{\mathfrak{G}}$ by the condition $\alpha(\mathfrak{x} + k\mathfrak{g}) = \alpha\mathfrak{x} + k\mathfrak{g}(\alpha)$ and let us, moreover, consider that in the group $\bar{\mathfrak{G}}$ there exist relations such that two of its elements coincide if and only if, they give the same homomorphisms of X in K. By virtue of these conditions the groups X and $\bar{\mathfrak{G}}$ are orthogonal, and by the first part of the theorem already proved X is the group of characters for $\bar{\mathfrak{G}}$; but, since $(X, \mathfrak{G}) = (0)$, $\mathfrak{G} = \bar{\mathfrak{G}}$; that is, the homomorphism $\mathfrak{g}$ is already contained among the homomorphisms of the group given by the elements of the group $\mathfrak{G}$.

LEMMA 8. *Let* X *and* $\mathfrak{G}$ *be orthogonal groups,* $\mathfrak{x}$ *an element of* $\mathfrak{G}$ *of order* g ($g = 0$ *or* $g > 1$). *If* $\gamma \subset K$ *and has an order which is a divisor of* g, *then there exists an* $\alpha \subset X$, *such that* $\alpha\mathfrak{x} = \gamma$.

PROOF. Let $\mathfrak{X}$ be the cyclic group with generator $\mathfrak{x}$. Evidently there exists a homomorphism β of the group $\mathfrak{X}$ in K such that $\beta(\mathfrak{x}) = \gamma$. By Lemma 1 the homomorphism β can be extended into a homomorphism $\bar{\alpha}$ of the entire group $\mathfrak{G}$ in K. By Theorem 5 there exists in X such an element α that $\bar{\alpha}(\mathfrak{y}) = \alpha(\mathfrak{y})$, for each $\mathfrak{y} \subset \mathfrak{G}$, that is, $\alpha\mathfrak{x} = \gamma$.

REMARK 5. If the group $\mathfrak{G}$ has a system of n linearly independent generators $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n$, then the group X of its characters has a particularly simple structure. In fact it decomposes into a direct sum of n continuous cyclic groups. Topologically, the group X is in this case the product of n circles, that is, an n-dimensional torus and we shall therefore call it a toroidal group. For $n = 1$, this is almost obvious. In fact, let $\alpha \subset X$, then $\alpha(\mathfrak{x}_1) = \gamma \subset K$ but for a given γ, this determines α uniquely since $\mathfrak{x}_1$ is a generator of $\mathfrak{G}$. In this way X and K are isomorphic. For a general n, each of the generators $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n$ can be mapped on K entirely arbitrarily and we obtain n continuous cyclic summands.

II. Structure of a Compact Group

In this chapter I make use of the results of F. Peter and H. Weyl,[3] which have become applicable to any compact group satisfying the second axiom of countability, thanks to the results of Haar.[4] I show that every compact commutative group satisfying the second axiom of countability is the group of characters of some discrete group.[7] Thus all the results of the preceding chapter become applicable to arbitrary compact, commutative groups.

All the continuous groups considered here are compact. The group is written multiplicatively.

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[7] To understand my paper it is sufficient to know the paper of Haar (Note 4).

5. In order to obtain the above mentioned result I formulate those results of Peter and Weyl, which I shall use.

These are as follows:

If Ω is a continuous compact group satisfying the second axiom of countability, then there exists on it a finite or denumerable system

$$f_{r,k_r}(\xi), \qquad (r = 1, 2, \cdots ; k_r = 1, 2, \cdots , n_r)$$

of real numerical continuous functions depending on the argument ξ, which is an element of the group Ω. Moreover the following conditions are satisfied:

1) For any element α of the group Ω

$$f_{r,i}(\xi \cdot \alpha) \equiv \sum_{j=1}^{n_r} m^j_{r,i}(\alpha) f_{r,j}(\xi), \qquad (i = 1, 2, \cdots , n_r)$$

where $m^j_{r,i}(\alpha)$ are real numbers forming an orthogonal matrix $M_r(\alpha)$ which depends continuously on α, and the number of whose rows and columns is equal to n_r.

2) If ϵ is the unit of the group Ω, then for every element α such that $\epsilon \neq \alpha \subset \Omega$ there exists a function $f_{r,i}(\xi)$ of our system such that $f_{r,i}(\xi \cdot \alpha) \not\equiv f_{r,i}(\xi)$, that is, there exists an r such that the matrix $M_r(\alpha)$ is not a unit matrix. It is easy to see that if α and β are two elements of Ω, then $M_r(\alpha \cdot \beta) = M_r(\beta) \cdot M_r(\alpha)$, where on the right we have matrix multiplication. Hence if the group Ω is commutative, then all the matrices $M_r(\alpha)$ are commutative among themselves for a fixed r.

In order to apply the above results to commutative groups, I will have to consider orthogonal matrices in greater detail. In what follows it is not fruitful to confine ourselves to matrices with real elements, but it is better to make use of orthogonal matrices with complex elements. As is well known, a matrix M with complex elements is said to be orthogonal if the following condition is fulfilled. Let M' be the matrix obtained from M by transposing rows and columns, and let $\bar{M}'$ be the complex conjugate of M', then $M^{-1} = \bar{M}'$. It is easy to see that if M and N are two orthogonal matrices with complex elements, then their product MN is also orthogonal. It is further known that if M is an orthogonal matrix, then there exists an orthogonal matrix T such that TMT^{-1} is a diagonal matrix, all of whose diagonal elements have absolute value unity. Making use of this fact we can prove the following lemmas:

LEMMA 9. *Let α run over the elements of any set, and let $M(\alpha)$ be a system of orthogonal n-rowed matrices, any two of which are commutative with each other, then there exists an n-rowed orthogonal matrix S, such that for any α, $SM(\alpha)S^{-1}$ is a diagonal matrix all of whose diagonal elements have absolute value unity.*

PROOF. Let $M(\alpha_1)$ be a non-diagonal matrix of our system and let T_1 be an orthogonal matrix such that $T_1M(\alpha_1)T_1^{-1}$ is diagonal. Let us designate its diagonal elements by e_i, $i = 1, 2, \cdots , n$, where the e_i are arranged in such a way that all equal ones are consecutive. Let us further denote by $M_1(\alpha)$ the matrix

$T_1M(\alpha)T_1^{-1}$. Its elements we designate by $m_i^j(\alpha)$. Since the matrices $M(\alpha_1)$ and $M(\alpha)$ are commutative, it follows that the matrices $M_1(\alpha_1)$ and $M_1(\alpha)$ have the same property. In this way we have $e_i m_i^j(\alpha) = m_i^j(\alpha) e_j$, that is $m_i^j(\alpha) = 0$ whenever $e_i \neq e_j$. It follows that the matrix $M_1(\alpha)$ decomposes into a system of square matrices along the diagonal of $M_1(\alpha)$, whose number s is not less than the number of distinct elements e_i. The number of distinct e_i, however exceeds unity, since if all e_i were equal, the matrix $M(\alpha_1)$ would be diagonal, which is contrary to hypothesis.

Let $M_1(\alpha_2)$ be a non-diagonal matrix. Transforming separately every square matrix of which it is composed, we obtain the general transforming matrix T_2, which is such that $T_2M_1(\alpha_2)T_2^{-1}$ is a diagonal matrix, while $T_2M_1(\alpha_1)T_2^{-1} = M_1(\alpha)$. Further, let us denote by $M_2(\alpha)$ the matrix $T_2M_1(\alpha)T_2^{-1}$. Writing out the conditions that $M_2(\alpha)$ and $M_2(\alpha_2)$ commute, we see as before that the matrix $M_2(\alpha)$ decomposes into separate square matrices, lying along its diagonal, whose number must now exceed s. Continuing this process we will reduce all our matrices to a diagonal form in a finite number of steps.

LEMMA 10. *Let Ω be a commutative compact group; then there exists a finite or denumerable system of continuous complex valued functions $\mathfrak{g}_n(\xi)$, $n = 1, 2, \cdots$, where the argument ξ is an element of the group Ω, having the following properties:*

1) *For any element α of Ω, $\mathfrak{g}_n(\xi\alpha) = e_n(\alpha)\,\mathfrak{g}_n(\xi)$, where $e_n(\alpha)$ is a complex number of absolute value unity, and $e_n(\alpha^{-1}) = (e_n(\alpha))^{-1}$.*

2) *For any element α of Ω with $\alpha = \epsilon$, there exists a number m such that $\mathfrak{g}_m(\xi\alpha) \not\equiv \mathfrak{g}_m(\xi)$, that is, $e_m(\alpha) \neq 1$.*

PROOF. Let us examine the system of functions $f_{r,i}(\xi)$ on Ω for a fixed r (which were mentioned in the beginning of this section) and the corresponding system of matrices $M_r(\alpha)$. Since the group Ω is commutative, the matrices $M_r(\alpha)$ also commute. Hence we can apply Lemma 9 to the system of matrices $M_r(\alpha)$ and there exists a transforming matrix S which reduces the matrices $M_r(\alpha)$ to diagonal form. We shall denote the elements of S by S_i^j. It is easy to see that the function

$$\sum_{j=1}^{n_r} S_i^j f_{r,j}(\xi), \qquad (i = 1, 2, \cdots, n_r)$$

satifies condition 1) of our lemma.

Applying the above operation to all the systems of functions $f_{r,i}(\xi)$ for every r, we obtain a totality of functions satisfying both conditions of the lemma.

FIRST FUNDAMENTAL THEOREM. *Let Ω be a continuous compact commutative group satisfying the second axiom of countability, and $\mathfrak{G}$ the discrete group of its characters, then the group Ω is isomorphic to the group of characters of $\mathfrak{G}$.* (See Definitions 1 and 1′.)

PROOF. Let $\mathfrak{g}_n(\xi)$, $(n = 1, 2, \cdots)$ be a system of functions defined on Ω and satisfying the conditions of Lemma 10. Let us show, first of all, that no func-

tion $\mathfrak{g}_n(\xi)$ can be equal to zero without being identically zero. Let us suppose that $\mathfrak{g}_n(\alpha) = 0$, then

$$\mathfrak{g}_n(\xi) = \mathfrak{g}_n(\alpha\alpha^{-1}\xi) = e_n(\alpha^{-1}\xi)\mathfrak{g}_n(\alpha) = 0.$$

We shall suppose that from the system $\mathfrak{g}_n(\xi)$ all identically zero functions have been removed. This does not violate the conditions of Lemma 10. Since none of the functions $\mathfrak{g}_n(\xi)$ now admit the value zero, they can be normalized in such a way that $\mathfrak{g}_n(\epsilon) = 1$, where ϵ is the unit of the group Ω. Let us suppose that this normalisation has been carried out. This, likewise will not affect the conditions of Lemma 10. We will have

$$\mathfrak{g}_n(\xi) = \mathfrak{g}_n(\epsilon\xi) = e_n(\xi)\mathfrak{g}_n(\epsilon) = e_n(\xi).$$

We now build a multiplicative group $\mathfrak{G}$ of functions defined on Ω, taking as generators the functions $\mathfrak{g}_n(\xi)$. Every element $\mathfrak{g}(\xi)$ of the group $\mathfrak{G}$ can be expressed in the form $(\mathfrak{g}_1(\xi))^{m_1} \cdot (\mathfrak{g}_2(\xi))^{m_2} \cdots (\mathfrak{g}_n(\xi))^{m_n}$ where n is sufficiently large and m_i is an integer. Since

$$\mathfrak{g}_n(\xi\eta^{-1}) = e_n(\eta^{-1})\mathfrak{g}_n(\xi) = \mathfrak{g}_n(\xi)(\mathfrak{g}_n(\eta))^{-1},$$

we have

$$\mathfrak{g}(\xi\eta^{-1}) = \mathfrak{g}(\xi)(\mathfrak{g}(\eta))^{-1}.$$

Let us further formulate the law of multiplication between the elements of the groups Ω and $\mathfrak{G}$. We shall define the product of $\alpha \subset \Omega$ by $\mathfrak{g}(\xi) \subset \mathfrak{G}$ as the element of the group K equal to

$$\frac{\log \mathfrak{g}(\alpha)}{2\pi i}.$$

From the conditions

$$\frac{\log \mathfrak{g}(\xi\eta^{-1})}{2\pi i} = \frac{\log \mathfrak{g}(\xi)}{2\pi i} - \frac{\log \mathfrak{g}(\eta)}{2\pi i}$$

and

$$\frac{\log [\mathfrak{g}'(\xi)(\mathfrak{g}''(\eta))^{-1}]}{2\pi i} = \frac{\log \mathfrak{g}'(\xi)}{2\pi i} - \frac{\log \mathfrak{g}''(\eta)}{2\pi i},$$

and also because of the continuity of the function $\mathfrak{g}_n(\xi)$, the groups Ω and $\mathfrak{G}$ form a couple (see Definition 3). Moreover the groups Ω and $\mathfrak{G}$ are orthogonal. In fact from condition 2 of Lemma 10, there is an $\alpha \neq \epsilon$ such that there is a $\mathfrak{g}(\xi)$ for which $\mathfrak{g}(\alpha) \neq 1$. Therefore we have $(\Omega, \mathfrak{G}) = (0)$. Moreover if $\mathfrak{g}(\xi)$ is not the unit of the group $\mathfrak{G}$, that is if $\mathfrak{g}(\xi)$ is not identically equal to unity, then there exists an element α, such that $\mathfrak{g}(\alpha) \neq 1$, and hence $(\mathfrak{G}, \Omega) = (0)$. It follows from Theorem 5 that each of the groups Ω and $\mathfrak{G}$ is the group of characters of the other.

III. Structure of a Locally Compact Connected Group

The purpose of this chapter is to prove the second Fundamental Theorem, that is to reduce the investigation of a locally compact, connected, commutative group, satisfying the second axiom of countability, to that of a compact group, already considered in Chapter II.

The groups are written additively, as in Chapter I.

6. We pass to the consideration of locally compact connected, but not necessarily compact groups by means of the following lemma:

LEMMA 11. *Let Ω be a locally compact, connected group. Then there exists a subgroup A of the group Ω, having not more than a denumerable number of elements, and having no limiting elements in Ω such that the factor group Ω/A is compact.*

In order to prove Lemma 11, we first establish:

LEMMA 12. *Let Ω be a non-compact group, satisfying the conditions of Lemma* 11. *Then there exists an element α in Ω which is such that the cyclic sub-group of Ω generated by α is free and has no limiting elements in Ω.*[8]

PROOF. a) Let U be some neighborhood of zero of the group Ω, whose closure $\bar{U}$ is compact. Let us denote by U_n the set of all the elements of Ω, which can be written in the form: $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ where $\alpha_i \subset U$, $(i = 1, 2, \cdots, n)$. Since U is a domain it follows that U_n is likewise a domain. It is easy to see that the closure $\bar{U}_n$ of the domain U_n is composed of all the elements of the form: $\beta_1 + \beta_2 + \cdots + \beta_n$ where $\beta_i \subset \bar{U}$, $(i = 1, 2, \cdots, n)$. Since $\bar{U}$ is compact, it follows easily that $\bar{U}_n$ is also compact. Since Ω is non-compact and connected, while $\bar{U}_n$ is compact, it follows that the boundary $U'_n = \bar{U}_n - U_n$ of the domain U_n is not empty.

b) Let $\alpha \subset \bar{U}_r$, $\beta \subset \bar{U}_s$ and $\gamma = \alpha + \beta \subset U'_{r+s}$. We will show that $\alpha \subset U'_r$, $\beta \subset U'_s$. Let us suppose on the contrary that α, for instance, belongs to U_r. Since β is a limiting element of U_s, an arbitrarily small element δ can be found, such that $\beta - \delta \subset U_s$. Since $U_r \supset \alpha$ is a domain, δ can be chosen sufficiently small so that $\alpha + \delta \subset U_r$. In this way we have $\gamma = (\alpha + \delta) + (\beta - \delta)$ where $\alpha + \delta \subset U_r$ and $\beta - \delta \subset U_s$. That is, from the construction of the domains U_n given in a), $\gamma \subset U_{r+s}$ which contradicts the assumption that $\gamma \subset U'_{r+s}$.

c) Let $\alpha_n \subset U'_n$. The set of elements α_n, $(n = 1, 2, \cdots)$ has no limiting points. Let us assume on the contrary that we have found two elements α_i and α_j, $i < j$ such that $\alpha_j - \alpha_i \subset U$. Then we would have $\alpha_j = \alpha_i + (\alpha_j - \alpha_i)$ that is, by b) $\alpha_j \subset U_{i+1}$, which is impossible since $\alpha_j \subset U'_j$, and U'_j does not intersect U_{i+1}.

d) Let us build up a system of elements ϵ_n $n = 1, 2, \cdots$, belonging to $\bar{U}$, and such that $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n \subset U'_n$. Let us choose an element $\gamma_n \subset U'_n$. Such an element exists since, by a), U'_n is not empty and can be represented in the form $\epsilon_{1,n} + \epsilon_{2,n} + \cdots + \epsilon_{n,n} = \gamma_n$ where $\epsilon_{i,n} \subset \bar{U}$, $(i = 1, 2, \cdots, n)$. Since $\bar{U}$ is compact, we can use the diagonal process to choose a sequence of numbers

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[8] Pontrjagin, *Über den algebraischen Inhalt topologischer Dualitätssätze*. Math. Ann. 105 (1931 pp. 165–205).

n_k, $(k = 1, 2, \cdots)$, such that $\lim_{k\to\infty} \epsilon_{i, nk}$ exists. Let us denote this limit by ϵ_i, $(i = 1, 2, \cdots)$. Since $\epsilon_{i,n} \subset \bar{U}$, $(i = 1, 2, \cdots, n)$, then $\epsilon_{1,n} + \epsilon_{2,n} + \cdots + \epsilon_{j,n} \subset \bar{U}_j$ and $\epsilon_{j+1,n} + \epsilon_{j+2,n} + \cdots + \epsilon_{n,n} \subset \bar{U}_{n-j}$, $j < n$ but $(\epsilon_{1,n} + \epsilon_{2,n} + \cdots + \epsilon_{j,n}) + (\epsilon_{j+1,n} + \epsilon_{j+2,n} + \cdots + \epsilon_{n,n}) \subset U'_n$. In this way using b) we have

$$\epsilon_{1,n} + \epsilon_{2,n} + \cdots + \epsilon_{j,n} \subset U'_j.$$

$$\lim_{k\to\infty} (\epsilon_{1,n_k} + \epsilon_{2,n_k} + \cdots + \epsilon_{j,n_k}) = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_j.$$

The left hand side of the last equation belongs to U'_j, since U'_j is closed. In this way the construction is completed.

e) Let us select a system of m elements from the system constructed in d) and let us denote these elements by $\epsilon_{n_1}, \epsilon_{n_2}, \cdots, \epsilon_{n_m}$, where we suppose that $n_i \neq n_j$ with $i \neq j$. Then $\epsilon_{n_1} + \epsilon_{n_2} + \cdots + \epsilon_{n_m} \subset U'_m$. In fact there exists an n sufficiently large so that $n > n_j$; $j = 1, 2, \cdots, m$. Let us split the sum $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ into two parts. The first part shall consist of $\epsilon_{n_1} + \epsilon_{n_2} + \cdots + \epsilon_{n_m}$ and the second part of all the remaining summands. The first part belongs to $\bar{U}_m$, and the second part to $\bar{U}_{n-m}$. Hence, from b) $\epsilon_{n_1} + \epsilon_{n_2} + \cdots + \epsilon_{n_m} \subset U'_m$.

Let α be a limiting element of the set ϵ_n. Then there exists a system $\epsilon_{n_1}, \epsilon_{n_2}, \cdots, \epsilon_{n_m}$, with an arbitrary m, in an arbitrarily small neighborhood of the element α. From this we conclude that for an arbitrary m, the element $m\alpha \subset U'_m$. Hence all the elements $m\alpha$ are distinct for distinct m's and have no limiting elements in Ω by c). Hence Lemma 12 is proved.

Proof of Lemma 11. Let U be an arbitrary neighborhood of zero of the group Ω, whose closure $\bar{U}$ is compact. Let us construct an increasing, transfinite sequence of subgroups A_i of the group Ω, each of which has only zero belonging to $\bar{U}$. Let A_1 be the group which contains only zero. Let us suppose further that all the groups A_i for $i < j$ have already been constructed. If j has no predecessor, we take for A_j the sum of all A_i with $i < j$. The group thus obtained has only zero in common with $\bar{U}$. Now let $j = k + 1$. Since A_k has only zero in common with $\bar{U}$, it follows A_k has no limiting points in Ω, and, hence, A_k is closed. We can now consider the factor group Ω/A_k which we denote by Ω', and the set $\bar{U}'$ which is obtained from $\bar{U}$ in forming the factor group. It is obvious that $\bar{U}'$ is compact. If Ω' is a compact group then we will stop the construction of the sequences A_i at the number k. If Ω' is not compact, we will apply to it Lemma 12. Let α be an element of the group Ω' constructed in Lemma 12. Since $\bar{U}'$ is compact, there exists only a finite number of elements of the form $m\alpha$, where m is an integer, which belong to $\bar{U}'$. For in the contrary case the set of elements of the form $m\alpha$ would have a limiting element in $\bar{U}'$. Hence there exists a sufficiently large number n, such that all the elements of the form $m\alpha$ with $|m| > n$ lie outside $\bar{U}'$. We denote the subgroup with the generator $n\alpha$ by A'. The complete image of the subgroup A' in the group Ω we denote by A_{k+1}. It is easy to see that A_{k+1} does not coincide with A_k, and has only zero in common with $\bar{U}$.

Since Ω satisfies the second axiom of countability and is locally compact, it can be covered with a countable system of domains, the closure of each of which is compact. Consequently every non-denumerable subset of the group Ω, must have a limiting point in Ω. But A_i has no limiting points in Ω, hence A_i is denumerable. That is, our construction can be stopped at a countable step, and we obtain in the end, a group A which satisfies the conditions of Lemma 11.

7. DEFINITION 4. *We will call the group of vectors in the n-dimensional affine space, an n-dimensional vector group. This group decomposes into a direct sum of subgroups, isomorphic to the additive group of real numbers.*

LEMMA 13. *Let* T *be a connected group, having a subgroup* A, *which has no limiting points in* T, *and which is such that the factor group* $T/A = T'$ *is a toroidal group (see Remark* 5), *then* T *is the direct sum of a certain vector group* M *and a toroidal group* Λ.

PROOF. Let r be the dimensionality of T'. Let us denote by N, an r-dimensional vector group, and let B be that subgroup of N all of whose elements have integral coordinates in some coordinate system. It is easy to see that T' is isomorphic to the factor group N/B. Hence using Schreier's[9] result, N covers universally the groups T' and T, and hence is isomorphic to the factor group N/B', where B' is a subgroup of B. By an integer transformation of the coordinates of N we can obtain a system of generators $\beta_1, \beta_2, \cdots, \beta_k$, $k \leqq r$ of B', such that β_i has all its coordinates equal to zero, except the i-th one, which is equal to some integer m_i. From this it is easy to conclude that T is isomorphic to a direct sum of a k-dimensional toroidal group and an $(r - k)$-dimensional vector group.

LEMMA 14. *Let* Ω *be a locally compact, connected group, and* U *an arbitrary neighborhood of zero in* Ω. *Then there exists a compact subgroup* Λ *situated in* U, *and such that the factor group* Ω/Λ *decomposes into a direct sum of a vector group and a toroidal group.*

PROOF. Let A be the subgroup of the group Ω, constructed in Lemma 11. Since A has no limiting element in Ω, there exists a sufficiently small neighborhood of zero V in Ω, such that if α and β are two elements of V with $\alpha - \beta \subset A$, then $\alpha = \beta$. Let us also suppose that $V \subset U$.

Since Ω is connected, the compact factor group $\Omega' = \Omega/A$ is also connected. From the First Fundamental Theorem it follows that the group Ω' is the group of characters of a certain discrete group $\mathfrak{G}$. Since Ω' is connected, $\mathfrak{G}$ contains no elements of finite order. (See Appendix 2, Theorem 1c.) Let $\mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_n$ be an increasing sequence of subgroups of the group $\mathfrak{G}$, such that each of the subgroups has a finite number of generators, and the sum of the subgroups coincides with $\mathfrak{G}$. Let $\Phi_n = (\Omega', \mathfrak{H}_n)$. Obviously Φ_n forms a decreasing sequence of subgroups, whose intersection contains only zero. Let V' be the image of V in forming the factor group Ω/A. From the construction, V is reflected on V'

[9] O. Schreier: *Die Verwandlschaft stetiger Gruppen im Grossen.* Abh. Math. Seminar der Hamburg Univ. 5 (1926–7) 233–244.

in a one to one way and V' forms a neighborhood of zero in Ω'. Since the Φ_n intersect in the zero of Ω', there exists a sufficiently large number m, such that $\Phi_m \subset V'$. The factor group Ω'/Φ_m is the group of characters of the group $\mathfrak{H}_m$ (see Theorem 3). Since the group $\mathfrak{H}_m$ has no elements of finite order and has a finite number of generators, it follows that $\mathfrak{H}_m$ has a finite system of linearly-independent generators, that is, the group $\Omega'/\Phi_m = \mathrm{T}'$ is toroidal (see Remark 5).

Let Γ be that portion of the image of the group Φ_m in the group Ω, which lies entirely in V. Since the mapping of V on V' is single valued, the group Γ is isomorphic to Φ_m. Since $\Gamma \subset V$, the intersection of Γ and A is a zero of the group Ω. Let $\Omega'' = \Omega/\Gamma$, and let A' be the subgroup of the group Ω' into which A is mapped. It is easy to see that A' has no limiting elements in Ω'', and that Ω''/A' is isomorphic to T'. Since T' is toroidal, it follows from Lemma 13 that Ω'' is a direct sum of a vector group and a toroidal group.

LEMMA 15. *Let Ω be a locally compact connected group. Then it contains a maximal compact subgroup Δ (that is, Δ is such that every other compact subgroup of Ω is a subgroup of Δ), and the factor group Ω/Δ is a vector group.*

PROOF. Let U be a neighborhood of zero in Ω, whose closure is compact, and let Γ be the subgroup constructed in Lemma 14. By the same lemma the factor-group Ω/Γ decomposes into the direct sum of the toroidal group Λ, and the vector group M. Let Δ be the complete image of the group Λ in the group Ω. Since Γ and Λ are compact, Δ is also compact and the factor-group Ω/Δ is isomorphic to M and is a vector group. Let us suppose that there exists a subgroup Δ', which is compact, but which does not belong to Δ. Let $\alpha \subset \Delta'$ and $\alpha \not\subset \Delta$. Let α' be the element into which α goes in forming the factor-group Ω/Δ. Since $\alpha \not\subset \Delta$, $\alpha \neq 0$, this means that the sequence of multiples of the element α' has no limiting elements in Ω/Δ. But this is impossible since all the multiples of α belong to Δ' and their set must have a limiting element.

THE SECOND FUNDAMENTAL THEOREM: *A locally compact connected group Ω satisfying the second axiom of countability decomposes into a direct sum of a compact subgroup Δ, and a vector subgroup* N (*see Definition* 4), *where the subgroup Δ is determined uniquely, and the demensionality r of the group* N *is an invariant of the group Ω.*

PROOF. Let Δ be the maximal compact subgroup of the group Ω, constructed in Lemma 15. Let us choose a decreasing sequence $U_1, U_2, \cdots, U_n, \cdots$ of neighborhoods of zero of the group Ω, whose closures are compact, and whose intersection contains only zero. Let us construct the sequence $\Omega_0 = \Omega, \Omega_1, \Omega_2, \cdots, \Omega_n, \cdots$ of closed, connected subgroups of the group Ω, such that the intersection Γ_n of the group Ω_n and the group Δ is contained in U_n, and such that every element of the group Ω can be written in the form $\alpha + \beta$, where $\alpha \subset \Delta$, and $\beta \subset \Omega_n$.

Let us suppose that Ω_n is already constructed. It is easy to see that Γ_n is the maximal, compact subgroup of the group Ω_n. In fact every compact subgroup of Ω_n belong to Δ by Lemma 15, and hence to the intersection of Δ and Ω_n, which

is Γ_n. The factor-group Ω_n/Γ_n is isomorphic to Ω/Δ, and hence the dimensionality of Ω_n/Γ_n is r. By Lemma 14, there exists a subgroup $\Gamma'_{n+1} \subset U_{n+1}$ of the group Ω_n which is such that Ω_n/Γ'_{n+1} decomposes into a direct sum of a toroidal subgroup Λ and a vector subgroup M. Since Λ is the maximal compact subgroup of Ω_n/Γ'_{n+1}, the subgroup Γ_n is the complete image of Λ in Ω_n. Let us denote by Ω'_{n+1} the complete image of the group M in Ω_n. Since the intersection of Λ and M in the group Ω_n/Γ'_{n+1} is zero, the intersection of Ω'_{n+1} and Γ_n is Γ'_{n+1}. Hence the intersection of Ω'_{n+1} and Δ is Γ'_{n+1}. Let Ω_{n+1} be the zero component of the group Ω'_{n+1}; such a group is, as is well known, a closed subgroup of the group Ω'_{n+1}. We denote by Γ_{n+1} the intersection of Ω_{n+1} and Γ'_{n+1}, then the intersection of Ω_{n+1} and Δ is Γ_{n+1}. We have then Ω/Δ is isomorphic to Ω_n/Γ_n which is isomorphic to $\Omega'_{n+1}/\Gamma'_{n+1}$ which in turn is isomorphic to $\Omega_{n+1}/\Gamma_{n+1}$. Hence the latter factor-group is an r-dimensional vector space. In forming the factor, group Ω/Δ, the subgroup Ω_{n+1} goes over into a subgroup isomorphic to $\Omega_{n+1}/\Gamma_{n+1}$, but since Ω/Δ and $\Omega_{n+1}/\Gamma_{n+1}$ are vector groups of the same dimensionality, $\Omega_{n+1}/\Gamma_{n+1}$ cannot be a proper subgroup of the group Ω/Δ. Hence, in forming the factor-group Ω/Δ, the subgroup Ω_{n+1} goes over into the whole group Ω/Δ. In this way every element of Ω can be represented in the form $\alpha + \beta$, where $\alpha \subset \Delta$ and $\beta \subset \Omega_{n+1}$.

This last property is also true for the intersection N of all the groups Ω_n, but the intersection of N and Δ contains only zero. Hence Ω is the direct sum of the subgroups Δ and N, with N isomorphic to Ω/N.

IV. Structure of a Locally Connected Group

The purpose of the present chapter is to investigate the structure of a connected, locally connected, locally compact group, satisfying the second axiom of countability. We solve the problem in the Third Fundamental Theorem, which shows that a group of the above type reduces to a direct sum of a finite or infinite number of continuous cyclic groups and a finite number of groups isomorphic to the additive group of real numbers. In the case of compact groups, the latter summands do not exist and we get only one compact, connected and locally connected group for every finite or infinite dimensionality.

A space is said to be locally connected if any two of its points which are sufficiently close together can be connected by a short path. Hence a locally Euclidean space is a locally connected finite-dimensional space, so that the Third Fundamental Theorem solves the question of the structure of a locally Euclidean group, as a special case.

8. Lemma 16. *Let $\mathfrak{G}$ be a discrete group which has no elements of finite order, and which is such that any increasing sequence of subgroups $\mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_n, \cdots$ of the same finite rank stops. Then there exists a finite or infinite system of linearly independent generators of the group $\mathfrak{G}$.*

Proof. Since every increasing sequence of subgroups of $\mathfrak{G}$ of a fixed finite

rank stops, every subgroup of finite rank can be included in the maximal subgroup of the same rank, having a finite number of generators.

Let us construct an increasing sequence $\mathfrak{G}_1, \mathfrak{G}_2, \cdots, \mathfrak{G}_n, \cdots$ of subgroups of $\mathfrak{G}$, such that $\mathfrak{G}_n$ is the maximal subgroup of rank n having a finite number of generators (this means that $\mathfrak{G}_n$ cannot be included in any subgroup of the same rank). Moreover the sum of all the groups $\mathfrak{G}_n$ coincides with $\mathfrak{G}$.

Let $\xi_1, \xi_2, \cdots, \xi_m, \cdots$ be a numbered sequence of all the elements of the group $\mathfrak{G}$. Let $\mathfrak{G}_1$ be the maximal subgroup of $\mathfrak{G}$ of rank 1, containing ξ_1. Let us suppose that $\mathfrak{G}_n$ is already constructed. Let m be the least number such that $x_m \notin \mathfrak{G}_n$, and let the group $\mathfrak{G}_{n+1}$ be the maximal subgroup of rank $n+1$ containing $\mathfrak{G}_n$ and x_m. Let us construct a sequence of elements $\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_n, \cdots$ of the group $\mathfrak{G}$, which is such that the system $\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_n$ forms a linearly independent basis of the group $\mathfrak{G}_n$. Since $\mathfrak{G}$ has no elements of finite order, the subgroup $\mathfrak{G}_1$, having a finite number of generators, is a free cyclic group. Let us denote its generator by $\mathfrak{g}_1$. Let us show that the factor-group $\mathfrak{G}_{n+1}/\mathfrak{G}_n$ is cyclic. Since $\mathfrak{G}_{n+1}/\mathfrak{G}_n$ has a finite number of generators and is of rank 1, it will suffice to show that $\mathfrak{G}_{n+1}/\mathfrak{G}_n$ has no elements of finite order. Let us suppose that there exists a subgroup $\mathfrak{H}$ of finite order, of the group $\mathfrak{G}_{n+1}/\mathfrak{G}_n$. Then the image of the group $\mathfrak{H}$ in $\mathfrak{G}_{n+1}$ is a group of rank n containing $\mathfrak{G}_n$, which is impossible since $\mathfrak{G}_n$ is a maximal subgroup. Hence $\mathfrak{G}_{n+1}/\mathfrak{G}_n$ has a generator $\mathfrak{g}'_{n+1}$. Let us designate one of the images of $\mathfrak{g}'_{n+1}$ in $\mathfrak{G}_{n+1}$ by $\mathfrak{g}_{n+1}$. The system $\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_n, \cdots$ thus obtained is a linearly independent system of generators of the group.

LEMMA 17. *Let $\mathfrak{G}$ be a group of finite rank r, having no elements of finite order. Let $\mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_n, \cdots$ be an increasing infinite sequence of subgroups of rank r of the group $\mathfrak{G}$, which are such that each $\mathfrak{H}_n$ has a finite number of generators and the sum of all $\mathfrak{H}_n$ coincides with $\mathfrak{G}$. Then the group of characters X of the group $\mathfrak{G}$ cannot be locally connected.*

PROOF. Let $\Phi_n = (X, \mathfrak{H}_n)$ and $T_n = X/\Phi_n$. T_n is isomorphic to the group of characters of $\mathfrak{H}_n$, and since $\mathfrak{H}_n$ has a finite number of generators, and no elements of finite order, it must have a finite system of linearly independent generators, that is T_n is a toroidal group (see Remark 5). Let U be a neighborhood of zero in T_1, homomorphic to a sphere. Such a neighborhood always exists since T_1 is a manifold. Let V be a sufficiently small neighborhood of zero in X, such that its image on reflection in X/Φ_1 lies completely in U. Since the sequence $\mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_n \cdots$ is infinite, the group $\mathfrak{G}/\mathfrak{H}_1$ is infinite and hence the group of its characters Φ_1 (see Theorem 3) is also infinite. But Φ_1 is compact and hence has no isolated elements. In this way there exists an element α in Φ_1, which is arbitrarily close to zero. Let us suppose that X is locally connected and let α be sufficiently close to zero so that α and zero can be connected by a continuous path $l \subset V$. Since the sum of all $\mathfrak{H}_n$ coincides with $\mathfrak{G}$, the intersection of all Φ_n coincides with zero and hence there exists a sufficiently large number m such that $\alpha \notin \Phi_m$. Let Φ'_1, α' and l' be the images of Φ_1, α and l in T_m with a homomorphism X/Φ_m, $\alpha' \neq 0$. Since $X/\Phi_1 = T_1$, $T_m/\Phi'_1 = T_1$. Let us denote by l'' the image of l' in T_1 with the homomorphism T_m/Φ'_1.

Since $l \subset V$, $l'' \subset U$ and hence the closed path l'' in $\mathbf{T}_1$ is homotopic to zero since U is homomorphic to a sphere. But this is impossible, because, using Schreier's[9] results, $\mathbf{T}_m$ covers $\mathbf{T}_1$, and the open path l' in $\mathbf{T}_m$ cannot go over into a closed path l'', homotopic to zero in $\mathbf{T}_1$.

THEOREM 6. *The connected, locally connected, compact group* X *satisfying the second axiom of countability decomposes into a direct sum of a finite or denumerable number of continuous cyclic groups.* (See Appendix 1 for the notion of a direct sum.)

PROOF. From the First Fundamental Theorem, X is the group of characters of a certain group $\mathfrak{G}$. Since X is connected, $\mathfrak{G}$ does not contain any elements of finite order (see Appendix 2, Theorem 1c). If the group $\mathfrak{G}$ satisfies the conditions of Lemma 16, it decomposes into a direct sum of free cyclic groups and hence by Theorem 1a of Appendix 1, X decomposes into a direct sum of continuous cyclic groups.

Let us now suppose that $\mathfrak{G}$ does not satisfy the conditions of Lemma 16, then there exists in $\mathfrak{G}$ an increasing infinite sequence $\mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_n, \cdots$ of groups of rank r, each of which has a finite system of generators. Let $\mathfrak{H}$ be the sum of all the groups $\mathfrak{H}_n$. The rank of $\mathfrak{H}$ is r. Let $\Phi = (\mathrm{X}, \mathfrak{H})$. Since X is locally connected, its continuous image X/Φ has the same property. But by Theorem 3, the group X/Φ is the group of characters of the group $\mathfrak{H}$, which contradicts Lemma 17.

THIRD FUNDAMENTAL THEOREM. *A connected, locally connected, locally compact group* Ω, *satisfying the second axiom of countability, decomposes into the direct sum of a finite or denumerable number of continuous cyclic groups and a vector group.*

PROOF. By the Second Fundamental Theorem, the group Ω decomposes into the direct sum of the compact group X and the vector group N. It is easy to see that X is isomorphic to Ω/N. Hence X is locally connected and connected and decomposes into the direct sum of continuous cyclic groups by Theorem 6.

COROLLARY. *By imposing on the Third Fundamental Theorem the further condition that* Ω *is of finite dimensionality, or the condition of shrinkage to zero of every sufficiently small closed contour, we obtain only a finite number of cyclic summands.*

PROOF. The first statement is obvious, the second follows from the fact that the presence of an infinite number of cyclic summands enables us to choose from among them an arbitrarily small subgroup, which is itself a closed contour but cannot shrink to a point, since it is a direct summand.

Appendix I

On the Decomposition of Groups into direct Sums

Here I give the connection between the decomposition of a group $\mathfrak{G}$ into a direct sum, and that of the group X of its characters. This enables me to establish the connection between the decompositions into direct sums of discrete and continuous compact groups. This allows me to give a number of examples not

devoid of interest; namely I give examples which contradict the results of J. W. Alexander and L. W. Cohen,[10] St. Pietrkowsky,[11] D. v. Dantzig.[12] We recall first of all the concept of a direct sum.

DEFINITION 1a. Let $\mathfrak{G}$ be a group and $\{\mathfrak{U}_m\}$ a system, of any power, of subgroups of the group $\mathfrak{G}$. We shall say that the group $\mathfrak{G}$ is decomposed into a direct sum of the groups $\mathfrak{U}_m$, if a) the minimum subgroup, which contains all the groups $\mathfrak{U}_m$, coincides with $\mathfrak{G}$; b) let $\mathfrak{V}_i$ be a minimum subgroup which contains all the subgroups $\mathfrak{U}_m$ with the exception of one, $\mathfrak{U}_i$. Then the intersection of the groups $\mathfrak{V}_i$ and $\mathfrak{U}_i$ contains only zero.

REMARK 1a. The power of the set of groups $\mathfrak{U}_m$, which give the decomposition of the group $\mathfrak{G}$ into a direct sum, is denumerable or finite. In fact, from every group $\mathfrak{U}_m$ we can choose an element different from zero. It is easy to see that all these elements are distinct, and since, by a general assumption of this paper, the group $\mathfrak{G}$ contains not more than a denumerable number of elements, the statement is proved. Therefore we can suppose that $m = 1, 2, \cdots$.

REMARK 2a. Every element $\mathfrak{x} \subset \mathfrak{G}$ is decomposed uniquely into a sum $\mathfrak{x} = \sum_{m=1}^{n} \mathfrak{x}_m$, where $\mathfrak{x}_m \subset \mathfrak{U}_m$, and n is an arbitrary integer, but sufficiently large depending on $\mathfrak{x}$. The totality of all the sums of this form is a group, which contains all the groups $\mathfrak{U}_m$ and must therefore, by Definition 1a, coincide with $\mathfrak{G}$. Suppose that we have two decompositions of $\mathfrak{x}$ into a sum $\mathfrak{x} = \sum_{m=1}^{n} \mathfrak{x}_m = \sum_{m=1}^{n} \mathfrak{x}'_m$; from this we have $\mathfrak{x}'_i - \mathfrak{x}_i = \sum_{m \neq i} (\mathfrak{x}_m - \mathfrak{x}'_m)$. But the left hand side of this equality belongs to $\mathfrak{U}_i$ and the right hand side to $\mathfrak{V}_i$, hence by Definition 1a, $\mathfrak{x}'_i = \mathfrak{x}_i$ for every i. Hence the uniqueness of the decomposition is established.

DEFINITION 1b. Let X be a compact group, and let Φ_m be a system, of any power, of the subgroups of X. We shall say that X is decomposed into a direct sum of the groups Φ_m if a) the minimum closed subgroup which contains all the groups Φ_m coincides with X. b) Let Ψ_i be the minimum closed subgroup which contains all the groups Φ with the exception of one group Φ_i, then the intersection of Ψ_i and Φ_i contains only zero.

REMARK 1b. The power of the set of groups Φ_m, which give the decomposition of X into a direct sum, is denumerable or finite. To prove this let us suppose that the index m runs over a well ordered set of values. Let us denote by Ξ_j the minimum closed group which contains all the groups Φ_i for $i \geqq j$. It

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PROOF. From the First Fundamental Theorem, X is the group of characters of a certain group $\mathfrak{G}$. Since X is connected, $\mathfrak{G}$ does not contain any elements of finite order (see Appendix 2, Theorem 1c). If the group $\mathfrak{G}$ satisfies the conditions of Lemma 16, it decomposes into a direct sum of free cyclic groups and hence by Theorem 1a of Appendix 1, X decomposes into a direct sum of continuous cyclic groups.

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REMARK 1a. The power of the set of groups $\mathfrak{U}_m$, which give the decomposition of the group $\mathfrak{G}$ into a direct sum, is denumerable or finite. In fact, from every group $\mathfrak{U}_m$ we can choose an element different from zero. It is easy to see that all these elements are distinct, and since, by a general assumption of this paper, the group $\mathfrak{G}$ contains not more than a denumerable number of elements, the statement is proved. Therefore we can suppose that $m = 1, 2, \cdots$.

REMARK 2a. Every element $\mathfrak{x} \subset \mathfrak{G}$ is decomposed uniquely into a sum $\mathfrak{x} = \sum_{m=1}^{n} \mathfrak{x}_m$, where $\mathfrak{x}_m \subset \mathfrak{U}_m$, and n is an arbitrary integer, but sufficiently large depending on $\mathfrak{x}$. The totality of all the sums of this form is a group, which contains all the groups $\mathfrak{U}_m$ and must therefore, by Definition 1a, coincide with $\mathfrak{G}$. Suppose that we have two decompositions of $\mathfrak{x}$ into a sum $\mathfrak{x} = \sum_{m=1}^{n} \mathfrak{x}_m = \sum_{m=1}^{n} \mathfrak{x}'_m$; from this we have $\mathfrak{x}'_i - \mathfrak{x}_i = \sum_{m \neq i} (\mathfrak{x}_m - \mathfrak{x}'_m)$. But the left hand side of this equality belongs to $\mathfrak{U}_i$ and the right hand side to $\mathfrak{V}_i$, hence by Definition 1a, $\mathfrak{x}'_i = \mathfrak{x}_i$ for every i. Hence the uniqueness of the decomposition is established.

DEFINITION 1b. Let X be a compact group, and let Φ_m be a system, of any power, of the subgroups of X. We shall say that X is decomposed into a direct sum of the groups Φ_m if a) the minimum closed subgroup which contains all the groups Φ_m coincides with X. b) Let Ψ_i be the minimum closed subgroup which contains all the groups Φ with the exception of one group Φ_i, then the intersection of Ψ_i and Φ_i contains only zero.

REMARK 1b. The power of the set of groups Φ_m, which give the decomposition of X into a direct sum, is denumerable or finite. To prove this let us suppose that the index m runs over a well ordered set of values. Let us denote by Ξ_j the minimum closed group which contains all the groups Φ_i for $i \geqq j$. It

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is obvious that Ξ_i forms a well ordered descending sequence of groups, but from a general assumption of this paper, the second axiom of countability for X, it follows that the sequence Ξ_i stops at a finite or countable term. In this way we can say without loss of generality that m assumes integer values.

REMARK 2b. Every element $\alpha \subset X$ can be uniquely represented as an infinite sum $\alpha = \sum_{m=1}^{\infty} \alpha_m$ (the infinite sum has a meaning, since X is a continuous group). The totality of all the convergent sums of the above type is easily seen to form a compact subgroup of the group X, which contains all the Φ_m and therefore must, in accordance with Definition 1b, coincide with X. The uniqueness of decomposition is proved in the same way as in Remark 2a.

THEOREM 1a. *If $\mathfrak{G}$ and X are orthogonal (see Definition 3) and if $\mathfrak{G}$ is decomposed into a direct sum of groups $\mathfrak{U}_m$, then there exists a decomposition of the group X into a direct sum of groups Φ_m, which is such that $(X, \mathfrak{U}_m) \supset \Phi_n$ for $n \neq m$, and $\mathfrak{U}_m$ and Φ_m are orthogonal.*

THEOREM 1b. *If $\mathfrak{G}$ and X are orthogonal, and if Φ_m gives the decomposition of X into a direct sum, then there exists a decomposition of the group $\mathfrak{G}$ into subgroups $\mathfrak{U}_m$ under the same conditions as in Theorem* 1a.

In order to prove both these theorems we first prove two preparatory lemmas.

LEMMA A. *Let $\mathfrak{G}$ and X be orthogonal and $\mathfrak{H}_m$ an arbitrary system of subgroups of $\mathfrak{G}$, whose intersection we will designate by $\mathfrak{H}$. If $\Xi_m = (X, \mathfrak{H}_m)$ and if the minimum closed subgroup which contains all Ξ_m be designated by Ξ, then $\Xi = (X, \mathfrak{H})$.*

PROOF. It is easily seen that $(X, \mathfrak{H}) \supset \Xi_m$, moreover since $(X, \mathfrak{H})$ is closed, it contains Ξ. Let $\mathfrak{x} \subset \mathfrak{G}$, and $\mathfrak{x} \not\subset \mathfrak{H}$, then there exists an i, such that $\mathfrak{x} \not\subset \mathfrak{H}_i$, hence there exists an $\alpha \subset \Xi_i \subset \Xi$ such that $\alpha\mathfrak{x} \neq 0$, so that $(\mathfrak{G}, \Xi) = \mathfrak{H}$.

LEMMA B. This lemma is analogous to Lemma A with $\mathfrak{G}$ and X interchanged. Its proof is analogous to the proof of Lemma A.

PROOF OF THEOREM 1a. With the notation of Definitions 1a and 1b, let us suppose that $\Phi_m = (X, \mathfrak{B}_m)$ and prove that Φ_m give the desired decomposition of the group X. Let $\mathfrak{x} \subset \mathfrak{B}_i$, in accordance with Remark 2a, $\mathfrak{x} = \sum_{m=1}^{n} \mathfrak{x}_m$. Since $\mathfrak{x} \subset \mathfrak{B}_i$, $\mathfrak{x}_i = 0$. If $\mathfrak{x}$ belongs simultaneously to all the $\mathfrak{B}_m$, then $\mathfrak{x} = 0$: that is the intersection of all the $\mathfrak{B}_m$ contains only zero. From this it follows, from Lemma A that the product of all the Φ_m coincides with X. Analogously the intersection of all the $\mathfrak{B}_m$ with the exception of $\mathfrak{B}_i$ coincides with $\mathfrak{U}_i$, from which it follows by Lemma A that $\Psi_i = (X, \mathfrak{U}_i)$. Since the product of $\mathfrak{U}_i$ and $\mathfrak{B}_i$ coincides with $\mathfrak{G}$, it follows from Lemma B and Theorem 3 and 4 that the intersection of Φ_i and Ψ_i contains only zero. In this way, Φ_m give a decomposition of the group X into a direct sum. From $\Psi_i = (X, \mathfrak{U}_i)$ it is obvious that $(X, \mathfrak{U}_m) \supset \Phi_n$, $m \neq n$. We shall show now that $\mathfrak{U}_m$ and Φ_m are orthogonal. Let $0 \neq \mathfrak{x} \subset \mathfrak{U}_i$. Because of the orthogonality of $\mathfrak{G}$ and X there exists an $\alpha \subset X$ with $\alpha\mathfrak{x} \neq 0$. In accordance with Remark 2b we have $\alpha = \sum_{m=1}^{\infty} \alpha_m$ but $\alpha_m\mathfrak{x} = 0$ with $m \neq i$, hence

$\alpha_i \mathfrak{x} = \alpha \mathfrak{x} \neq 0$, that is $(\mathfrak{U}_i, \Phi_i) = (0)$. In the same way we can prove that $(\Phi_i, \mathfrak{U}_i) = (0)$.

The proof of Theorem 1b is entirely analogous.

REMARK 3ab. From Theorems 1a and 1b, there exists a complete duality between the decomposition into a direct sum of a discrete group and that of the group of its characters. Moreover the decomposition of the one group into *a* not further decomposible sum corresponds to the decomposition of the other group into a sum which is likewise not decomposible any further. This fact enables us, starting with examples for discrete groups, to arrive automatically at example for continuous groups. It also shows that the question of decomposition of continuous compact groups is equivalent to the question of decomposition of discrete groups. (See First Fundamental Theorem.)

EXAMPLE 1. H. Prüfer[13] gave an example of a group $\mathfrak{G}$, all of whose elements are of a finite order and which cannot be decomposed into a direct sum of indecomposible subgroups, although it admits of a decomposition into a sum of further decomposible subgroups. If we construct for $\mathfrak{G}$ its group of characters X we get an example of a compact, zero-dimensional group (see Appendix 2, Theorem 2c), which possesses the same properties of decomposibility. A more detailed study of the group X shows that it satisfies the conditions of the paper of Alexander and Cohen[10] and also the conditions of the paper by St. Pietrkowsky.[11] In this way it gives an example which contradicts the results of these papers, since these papers assert the decomposibility of groups satisfying certain given conditions into non-decomposible subgroups.

The group $\mathfrak{G}$ is given by its generators $\mathfrak{y}$, $\mathfrak{x}_i$, $(i = 1, 2, \cdots)$ and relations between them $p\,\mathfrak{y} = 0$, $p^i \mathfrak{x}_i = \mathfrak{y}$, where p is any prime number. It is easy to see that the order of the element $\mathfrak{x}_i$ is p^{i+1}, and that the order of the element $\mathfrak{y}$ is p.

Let us form the group of characters X of the group $\mathfrak{G}$ by means of its generators, β, α_i, $(i = 1, 2, \cdots)$ where β and α_i are determined by the conditions $\beta(\mathfrak{y}) = \frac{1}{p}$, $\beta(\mathfrak{x}_i) = \frac{1}{p^{i+1}}$, $\alpha_i(\mathfrak{y}) = 0$, $\alpha_i(\mathfrak{x}_j) = 0$ with $i \neq j$, $\alpha_i(\mathfrak{x}_i) = \frac{1}{p^i}$. The right hand sides of these equalities are numbers, which are considered as elements of a group K. (See Definition 1.) It is easily seen that every element of the group X can be expressed as an infinite sum $b\,\beta + \sum_{i=1}^{\infty} a_i \alpha_i$ with $0 \leqq b < p$ and $0 \leqq a < p^i$, since the conditions $p^i \alpha_i = 0$, and $p\,\beta = \sum_{i=1}^{\infty} \alpha_i$ are fulfilled. Also the k^{th} neighborhood of zero in X is determined by the condition $b = 0$, $\alpha_i = 0$, with $i \leqq k$. The group X which is determined by its generators and given relations, and which is the group of characters of $\mathfrak{G}$, obviously satisfies the conditions of Alexander and Cohen. It is not difficult to show that it likewise satisfies the conditions of St. Pietrkowsky.

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[13] H. Prüfer: *Untersuchungen über die Zerlegbarkeit der abzählbaren primären Abelschen Gruppen.* Math. Zeitschr. 17 (1923) pp. 35–61 (§18).

EXAMPLE 2. Let us construct a group $\mathfrak{G}$ without any elements of finite order, whose rank is two and let $\mathfrak{G}$ be such that it is not decomposible into a direct sum. The group of character X of the group $\mathfrak{G}$ will be a compact connected group of dimensionality 2, (see Appendix 2, Theorems 1c and 2c) which is likewise not decomposible into a direct sum (see Theorem 1b), which contradicts the statements of the paper of D. v. Dantzig.[12]

We will give the group $\mathfrak{G}$ by its generators $\mathfrak{y}$, $\mathfrak{x}_i$, $(i = 0, 1, 2, \cdots)$ and the relation,

$$(*) \qquad 2^{k_{i+1}}\mathfrak{x}_{i+1} = \mathfrak{x}_i + \mathfrak{y}, \qquad (i = 0, 1, 2, \cdots),$$

where k_i are positive integers satisfying only the condition that some k's are arbitrarily large.

It is easy to see that the element $\mathfrak{y}$ of the group $\mathfrak{G}$ is free, since from the relation (*) we cannot eliminate all the $\mathfrak{x}_i$ in such a way as to leave a relation in $\mathfrak{y}$ alone. Let us designate by $\mathfrak{H}$ the subgroup of the group $\mathfrak{G}$ with the generator $\mathfrak{y}$. In forming the factor group $\mathfrak{G}/\mathfrak{H}$ if we denote by $\mathfrak{x}_i'$ the element of $\mathfrak{G}/\mathfrak{H}$ into which $\mathfrak{x}_i$ goes, then $\mathfrak{x}_i'$ is the system of generators of $\mathfrak{G}/\mathfrak{H}$ with the relation $2^{k_{i+1}}\mathfrak{x}_{i+1}' = \mathfrak{x}_i'$, so that the group contains no elements of finite order. Since $\mathfrak{H}$ and $\mathfrak{G}/\mathfrak{H}$ do not contain any elements of finite order, the group $\mathfrak{G}$ itself has the same property.

Because of the absence in $\mathfrak{G}$ of elements of finite order, division in $\mathfrak{G}$ is unique, that is the equation $g\mathfrak{a} = \mathfrak{b}$, where $\mathfrak{a} \subset \mathfrak{G}$, $\mathfrak{b} \subset \mathfrak{G}$, and g an integer, has not more than one solution. Therefore in $\mathfrak{G}$ we can use fractional coefficients without fear.

Now we shall show that $\mathfrak{G}$ does not contain any infinitely divisible elements, that is for every element $\mathfrak{b}$, a sufficiently large integer k can be found such that if $q > k$ the equation $q\mathfrak{a} = \mathfrak{b}$ has no solution or, which is the same, there exists no element $\frac{1}{q}\cdot\mathfrak{b}$. From this the non-decomposibility of $\mathfrak{G}$ can be derived without much effort.

Making use of fractional coefficients we can express all the generators of $\mathfrak{G}$ by means of $\mathfrak{y}$ and $\mathfrak{x}_0 = \mathfrak{x}$. Simple calculations show, that

$$\mathfrak{x}_i = \frac{\mathfrak{x} + \mathfrak{y}(1 + 2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_1+k_2+\cdots+k_{i-1}})}{2^{k_1+k_2+\cdots+k_i}}.$$

In this way every element of the group $\mathfrak{G}$ can be written in the form $r\mathfrak{x} + s\mathfrak{y}$, where r and s are rational numbers, but of course not every expression of this type represents an element of $\mathfrak{G}$. In particular, if there are two elements $r\mathfrak{x} + s'\mathfrak{y}$ and $r\mathfrak{x} + s''\mathfrak{y}$, then $s' - s''$ is an integer: In fact $(s'' - s')\mathfrak{y}$ is an element of $\mathfrak{G}$ since it is the difference of two of its elements. But since $\mathfrak{G}/\mathfrak{H}$ has no elements of finite order, $s'' - s'$ must be an integer, for otherwise $\mathfrak{G}/\mathfrak{H}$ would have an element of finite order. Let us suppose now that some element $\mathfrak{b}$ admits of infinite division. From what we just said, $\mathfrak{b}$ cannot be a multiple of $\mathfrak{y}$. There-

fore in the transition to the factor group, $\mathfrak{b}$ will go into an element $\mathfrak{b}'$, but if $\mathfrak{b}$ admits an infinite division, so will $\mathfrak{b}'$. However, in $\mathfrak{G}/\mathfrak{H}$ infinite division can only take place by arbitrary high powers of 2. Therefore $\mathfrak{b}$ admits division by arbitrary high powers of 2. But if $\mathfrak{b}$ admits such a division, an arbitrary multiple of $\mathfrak{b}$ will have the same property, and therefore it would be possible to find an element $\bar{\mathfrak{b}} = m\mathfrak{x} + n\mathfrak{y}$ which admits a division by arbitrarily high powers of 2 with m and n integers. In this way $\mathfrak{G}$ contains an element $\dfrac{m\mathfrak{x} + n\mathfrak{y}}{2^{k_1+k_2+\cdots+k_j}} = \mathfrak{b}_j$ with an arbitrarily large j. Let us form the difference $m\mathfrak{x}_j - \mathfrak{b}_j$; it is equal to

$$\mathfrak{y}\,\frac{m(1 + 2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_1+k_2+\cdots+k_{j-1}}) - n}{2^{k_1+k_2+\cdots+k_j}} = \mathfrak{y}\cdot t.$$

From the preceding remarks the number t must be an integer. Let us show that this is impossible. First of all it is clear that with a sufficiently large j, it is positive. Let us show that for some arbitrarily large j, $t < 1$. To prove this statement let us estimate t. Replacing the sum which is in the numerator of t by a geometrical progression we find easily that

$$t < \frac{(|m| + |n|)2^{k_1+k_2+\cdots+k_{j-1}+1}}{2^{k_1+k_2+\cdots+k_j}} = \frac{2(|m| + |n|)}{2^{k_j}}$$

but as k_j takes on arbitrarily large values, it is true that for some arbitrarily large j we have $t < 1$. In this way $\mathfrak{G}$ does not contain any elements which admit infinite division.

Let us suppose now that $\mathfrak{G}$ is decomposible into a direct sum of two groups $\mathfrak{G}'$ and $\mathfrak{G}''$. The groups $\mathfrak{G}'$ and $\mathfrak{G}''$ have to be of rank 1 and without elements of finite order. Let us study the structure of such a group $\bar{\mathfrak{G}}$ without any elements of finite order and of rank 1. Let $\mathfrak{u}$ and $\mathfrak{v}$ be elements of $\bar{\mathfrak{G}}$. Since the rank of $\bar{\mathfrak{G}}$ is 1 we have $n\,\mathfrak{v} = m\,\mathfrak{u}$ that is $\mathfrak{v} = \dfrac{m}{n}\,\mathfrak{u}$. Putting every element $\mathfrak{v}$ in correspondence with a rational number $\dfrac{m}{n}$ we obtain an isomorphic mapping of the group $\bar{\mathfrak{G}}$ into a group of rational numbers, that is $\bar{\mathfrak{G}}$ is isomorphic with some subgroup of the group of rational numbers. But a subgroup of the group of rational numbers is either cyclic or else contains elements which allow infinite division. But since $\mathfrak{G}$ has no elements which admit infinite division, it follows that $\mathfrak{G}'$ and $\mathfrak{G}''$ will have to be cyclic, which is obviously impossible.

Appendix 2

Connectivity and Dimensionality of the Group of Characters

Here I endeavor to explain the simplest topological properties of the group of characters X of some discrete group $\mathfrak{G}$ and to establish a connection between these properties and the algebraic properties of the group $\mathfrak{G}$.

THEOREM 1c. *The group of characters* X *of the group* $\mathfrak{G}$ *is connected if and only if the group* $\mathfrak{G}$ *has no elements of finite order.*

PROOF. Let us suppose that $\mathfrak{G}$ has an element $\mathfrak{x}$ of finite order. Let us denote the group which has $\mathfrak{x}$ for a generator by $\mathfrak{H}$: this group is finite. Let $\Phi = (X, \mathfrak{H})$. By Theorem 3, the group X/Φ is the group of characters of the group $\mathfrak{H}$. Since $\mathfrak{H}$ is finite, its group of characters X/Φ is also finite, by Lemma 3. Hence X can be continuously mapped by means of the homomorphism X/Φ into a finite system of elements. Hence X is not connected.

Let us suppose X can be divided into two non-intersecting compact parts U and V. Let $\mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_n, \cdots$ be an increasing sequence of subgroups of the group $\mathfrak{G}$, whose sum coincides with $\mathfrak{G}$, and each of which has a finite number of generators. Let $\Phi_n = (X, \mathfrak{H}_n)$. The intersection of all Φ_n contains only zero, hence there exists a sufficiently large number m, such that the following condition is satisfied for the group Φ_m. If $\alpha \subset U$ and $\beta \subset V$, then $\beta - \alpha \notin \Phi_m$. Because of this condition, U and V go into two non-intersecting compact sets in forming the factor group X/Φ_m. Hence X/Φ_m is not connected. But X/Φ_m is the group of characters of $\mathfrak{H}_m$ with a finite system of generators, by Theorem 3. If $\mathfrak{H}_m$ had no elements of finite order, it would have a finite system of linearly independent generators, and X/Φ_m would be a toroidal, that is a connected, group (see Remark 5). Hence $\mathfrak{H}_m$ contains elements of finite order and so also does $\mathfrak{G}$.

COROLLARY 1c. *Let* $\mathfrak{G}$ *be a discrete group, and let* X *be a group of its characters. Let* $\mathfrak{H}$ *be the subgroup of* $\mathfrak{G}$ *composed of all the elements of finite order in* $\mathfrak{G}$, *and let* $\Phi = (X, \mathfrak{H})$. *Then* Φ *is the component of zero in* X.

PROOF. By Theorem 3, Φ is the group of characters of $\mathfrak{G}/\mathfrak{H}$, and since $\mathfrak{G}/\mathfrak{H}$ has no elements of finite order, Φ is connected, by Theorem 1c. At the same time, Φ is the maximal connected subgroup of $\mathfrak{G}$. For suppose that Φ is a proper subgroup of Φ', and let $\mathfrak{H}' = (\mathfrak{G}, \Phi')$. Then $\mathfrak{H}'$ is a proper subgroup of $\mathfrak{H}$ (see Theorem 4), and hence $\mathfrak{G}/\mathfrak{H}'$ has elements of finite order, which means that its group of characters Φ' is not connected.

THEOREM 2c. *If* $\mathfrak{G}$ *is a discrete group, and* X *the group of its characters, the dimensionality of* X *equals the rank of* $\mathfrak{G}$.

PROOF. Let us prove that the dimensionality of X cannot exceed the rank r of $\mathfrak{G}$. If r is infinite, the statement has no meaning. Suppose that r is finite. Let $\mathfrak{H}_1, \mathfrak{H}_2, \cdots, \mathfrak{H}_n, \cdots$ be an increasing sequence of subgroups of $\mathfrak{G}$, having the rank r. Let the sum of all the $\mathfrak{H}_n$ equal $\mathfrak{G}$, and let each $\mathfrak{H}_n$ have a finite number of generators. Let $\Phi_n = (X, \mathfrak{H}_n)$. The intersection of all Φ_n contains only zero, that is, there exist arbitrarily small subgroups among the groups Φ_n. Since $\mathfrak{H}_n$ has a finite number of generators, it decomposes into a direct sum of r free cyclic subgroups and a finite group. Hence the group of its characters X/Φ_n is a direct sum of a toroidal group of rank r and a finite group (see Appendix 1, Theorem 1a and Remark 5), that is, the dimensionality of X/Φ_n is r. But for a sufficiently large n, X/Φ_n can be obtained from X by an arbitrarily small displacement, which, as is well known, cannot lower the dimensionality. Hence the dimensionality of X does not exceed r.

Let us show now that the dimensionality of X is not less than the rank r of $\mathfrak{G}$. To do this, it is sufficient to show that the dimensionality of the component of zero in X is not less than the rank of $\mathfrak{G}$, that is, using the preceding corollary, it is sufficient to discuss only the case where $\mathfrak{G}$ has no elements of finite order. In this case there exists a finite or infinite system $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n, \cdots$ of linearly independent elements of $\mathfrak{G}$, which is such that every element of $\mathfrak{G}$ can be expressed uniquely as a finite linear form in $\mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_n, \cdots$ with rational coefficients. Let us now build a homomorphic mapping of the group $\mathfrak{G}$ into an additive group of real numbers, putting every element $\mathfrak{x}$ of the group $\mathfrak{G}$ into correspondence with a real number $\alpha(t_1, t_2, \cdots, t_n; \mathfrak{x})$ where $t_1, t_2, \cdots, t_n$ are real numbers upon which the homomorphism depends. This homomorphism is determined by the conditions $\alpha(t_1, t_2, \cdots, t_n; \mathfrak{x}_i) = t_i$ with $i = 1, 2, \cdots, n$ and $\alpha(t_1, t_2, \cdots, t_n; \mathfrak{x}_j) = 0$ with $j > n$. It is easy to see that from these conditions α is uniquely determined. Reducing the number $\alpha(t_1, t_2, \cdots, t_n; \mathfrak{x})$ modulo unity, we obtain the homomorphism of the group $\mathfrak{G}$ into the group K. It is easy to see that if $0 < t_i < 1$, $i = 1, 2, \cdots, n$, then to every system of values $t_1, t_2, \cdots, t_n$ corresponds its own special homomorphism. Hence there is a subset in X homomorphic to the interior of an n-dimensional cube, but n is an arbitrary number not exceeding the rank of $\mathfrak{G}$.

COROLLARY 2c. *The group of characters* X *of the discrete group* $\mathfrak{G}$ *is zero-dimensional if, and only if, all the elements of* $\mathfrak{G}$ *are of finite order, since in this case the rank of* $\mathfrak{G}$ *is zero.*

Appendix 3

(Added in proof May 14, 1934).

The reason why we do not introduce topological relations in definition I' of the character group $\mathfrak{G}$ of a compact group X is that in $\mathfrak{G}$ there cannot exist any convergent sequence. We must prove that if X and $\mathfrak{G}$ form an orthogonal pair $\mathfrak{G}$ cannot contain a sequence $\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_n, \cdots$ converging to zero. We will prove that this is impossible by constructing a character χ of $\mathfrak{G}$ such that $\chi\,\mathfrak{g}_n$ does not converge to 0.

Let $\mathfrak{G}_n$ be the subgroup of $\mathfrak{G}$ generated by $\mathfrak{g}_1, \cdots, \mathfrak{g}_n$ and $\mathfrak{G}'$ the subgroup generated by $\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_n, \cdots$. We suppose first that $\mathfrak{G}'$ does not have a finite basis. Then there are arbitrarily big numbers n such that $\mathfrak{g}_n$ is not contained in $\mathfrak{G}_{n-1}$. Suppose that $\chi(\mathfrak{G}_{n-1})$ has already been constructed. If $\mathfrak{g}_n$ is independent of $\mathfrak{G}_{n-1}$ then we put $\chi\,\mathfrak{g}_n = \frac{1}{2}$. If $\mathfrak{g}_n$ is linearly dependent on $\mathfrak{G}_{n-1}$ we take a relation $k\,\mathfrak{g}_n = \mathfrak{g}'$, $\mathfrak{g}' \subset \mathfrak{G}_{n-1}$, with lowest possible positive integer k. Then we put $\chi\,\mathfrak{g}_n = \dfrac{\chi\,\mathfrak{g}'}{k}$, taking the value of $\dfrac{\chi\,\mathfrak{g}'}{k}$ that is as far away from zero as possible. The character χ now defined over $\mathfrak{G}'$ can be extended over $\mathfrak{G}$. As we have for arbitrarily big values of n either $\chi\,\mathfrak{g}_n = \frac{1}{2}$ or $\chi\,\mathfrak{g}_n = \dfrac{\chi\,\mathfrak{g}'}{k}$ with $k > 1$ so that $\chi\,\mathfrak{g}_n$ is at least $\frac{1}{4}$ away from 0, we do not have $\lim \chi\,\mathfrak{g}_n = 0$.

Now we take the case where $\mathfrak{G}'$ possesses a finite basis $\mathfrak{a}_1, \cdots, \mathfrak{a}_p, \mathfrak{b}_1, \cdots, \mathfrak{b}_q$, where $\mathfrak{a}_1, \cdots \mathfrak{a}_p$, are linearly independent and $\mathfrak{b}_1, \cdots \mathfrak{b}_q$ have finite order. Putting

$$\mathfrak{g}_n = r_1^n \mathfrak{a}_1 + \ldots + r_p^n \mathfrak{a}_p + s_1^n \mathfrak{b}_1 + \cdots + \mathfrak{b}_q^n$$

we can suppose that $r_1^n \neq 0$ for arbitrarily high n. Then we make $\chi \mathfrak{a}_i = 0$, $(i = 2, \cdots, p)$, $\chi \mathfrak{b}_i = 0$, $(i = 1, \cdots, q)$ and $\chi \mathfrak{a}_1 = d$. It is easy to see that d can be chosen in such a way that $\lim_{n\to\infty} r_1^n d \neq 0$.

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ON THE CHARACTERS OF DISCRETE ABELIAN GROUPS

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1. Let A be an arbitrary discrete abelian group. We select any basis of A, form a set of fundamental relations in the base elements, and from these relations construct two (finite, infinite, or transfinite) matrices P and p which determine the group A completely, (§2). Next, we consider the character group B of A. In much the same way as before we select a "pseudo-basis" of B, form a set of fundamental relations in the elements of the pseudo basis, and from these relations construct two matrices P' and Q' determining B, (§3). The matrices P' and Q' turn out to be the transposed of the matrices P and Q respectively. Finally, we show that there is always a group C composed of characters of B and simply isomorphic with the original group A, (§4).

In a subsequent paper we shall develop some of the consequences of the above result. The character group B of A has a natural topological structure. If we assign this structure to the group B, then the group C will be shown to be the group of all *continuous* characters of B. We shall thus obtain a slight extension of a theorem derived by L. Pontrjagin for the case where the group A is countable.[1] We shall also bring out how the matrices P' and Q' make it possible to obtain a clear geometrical picture of the character group B of A, at least in the case where A is countable.

Our method of procedure will be closely related to one suggested by R. E. A. C. Paley and N. Wiener for obtaining the characters of discrete, abelian groups.[2]

2. We shall be dealing exclusively with abelian groups and shall, therefore, describe the resultant of two group elements a and a' by a sum, $a + a' = a' + a$. The resultant of any finite sequence of group elements will thus be expressible as a finite linear form in the group elements with integer coefficients.

Now, let A be any discrete, abelian group. By a *basis* of A we shall mean any set of elements of A such that a general element of A can be expressed *in one or more ways* as a finite linear combination of elements of the set with integer coefficients. Thus, for example, the set of all elements of A will be one possible

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[1] This is one of the results in Mr. Pontrjagin's paper *on the theory of topological commutative groups* (pp. 361-388 of this issue). A summary of Mr. Pontrjagin's paper in the form of a letter to Mr. Lefschetz was known to me at the time of writing the present note.

In a paper *on the homology groups of abstract spaces*; these Annals, Vol. 35 (1934), pp. 130–151, I have corrected an error in a paper by L. W. Cohen and me *on a classification of the homology groups of compact spaces*; these Annals, Vol. 33 (1932), pp. 538–566. This error was pointed out by Mr. Pontrjagin and is mentioned in Appendix I to his paper.

[2] Characters of Abelian Groups, Proc. of Nat. Acad., vol. 19 (1933), pp. 253–257.

basis of A. Given an arbitrary basis of A, there will, in general, be certain linear combinations L of the base elements which are not identically zero, yet which represent the zero element of A. Any subset of the linear forms L will be called a *fundamental set* provided the forms L can each be expressed as finite linear combinations of forms belonging to the subset with integer coefficients. Given an arbitrary basis of A, we can always find a fundamental set composed of linearly independent forms $L_i = 0$ in the following manner:

We arrange the base elements of A in a well ordered set. Then if a base element a of A has the property that no multiple of a is expressible as a finite linear combination of the base elements preceding it, we say that a is a base element of the *first kind*, otherwise that it is of the *second kind*. Next, we split up the set of base elements into two well ordered subsets

$$\alpha_1, \alpha_2, \cdots ; \qquad a_1, a_2, \cdots \tag{2:1}$$

composed of the base elements α_i and a_i of the first and second kinds respectively. No linear combination of elements of the first kind can vanish identically, but to each base element a_i of the second kind there corresponds at least one linear relation,

$$L_s = \sum_s p_{is}\,\alpha_s + \sum_s q_{is}\,a_s = 0 \tag{2:2}$$

$$(q_{ii} \neq 0; \qquad q_{is} = 0 \text{ for } i < s),$$

expressing the fact that some multiple q_{ii} of a_i is a linear combination of elements α_s and of elements a_s preceding a_i. If we choose the relations (2:2) in such a manner that the coefficients q_{ii} of their last terms are positive and numerically as small as possible, then the forms (2:2) will constitute the desired fundamental set. For they will clearly be linearly independent, since each contains a base element a_i not appearing in any of the preceding forms. Moreover, every linear form L representing the zero element,

$$L = \Sigma\, p_s\,\alpha_s + \Sigma\, q_s\,\alpha_s = 0 \tag{2:3}$$

$$(s \leq i, \qquad q_i \neq 0),$$

will be expressible as a finite linear combination of the forms L_s $(s \leq i)$, as we can prove in the following manner, by transfinite induction. We first observe that the last coefficient q_i of L must be exactly divisible by the last coefficient q_{ii} of L_i, $(q_i = \lambda q_{ii})$, since the form L_i was so chosen that its coefficient q_{ii} was numerically as small as possible. Consequently, we may write

$$L = \lambda L_i + L',$$

where L' is a form which either involves no base element a_s of the second kind, or only elements a_s preceding a_i. In the first case, the form L' must vanish identically; in the second, we may assume as the hypothesis of the induction that L' is expressible as a finite linear combination of the forms L_s, $(s < i)$. In either

case, therefore, the form L will be a finite linear combination of the forms L_s, $(s \leq i)$.

We can think of the coefficients p_{is} and q_{is} of (2:2) as the elements of two (finite or infinite) matrices P and Q respectively with well ordered rows and well ordered columns. The matrix P will be subject to the sole restriction that it have at most a finite number of non-vanishing elements p_{is} in each row. The matrix Q will be a "square" matrix with zeros above the main diagonal, with no zeros at all on the main diagonal, and with at most a finite number of elements in each row.

Let

$$a = \Sigma \xi_s \alpha_s + \Sigma x_s a_s \tag{2:4}$$

be an arbitrary group element of A. Then we shall speak of the integer coefficients

$$(\xi_1, \xi_2, \cdots ; \quad x_1, x_2, \cdots) \tag{2:5}$$

as the *coordinates* of the element a with respect to the given basis. Of course, the element a will, in general, have more than one set of coordinates, since the form (2:4) will only be determined modulo the forms (2:2).

3. *A character* b of A will be defined as any function of a variable element a of A satisfying the following two conditions:

(i) At each element a of A the value of the function $b(a)$ is a real number reduced modulo 1.

(ii) The function b is linear in a, in the sense that if a and a' are any two elements of A then we have

$$b(a + a') \equiv b(a) + b(a') \pmod 1.$$

The *negative* of a character b will be the character with the values $-b(a)$, modulo 1: the *sum* of two characters b and b' will be the character with the values $b + b'$. Thus the set of all characters b of A will form a group B under the operations of addition. The group B will be the *character group* of A.

We shall now show how to construct the character group B of A. Clearly, a character b of A will be completely determined when we know the values of b at all the base elements of A. For let

$$(\eta_1, \eta_2, \cdots ; \quad z_1, z_2, \cdots) \tag{3:1}$$

be any set of real numbers congruent, modulo 1, to the values of b at the base elements α_i and a_i respectively.

$$b(\alpha_i) \equiv \eta_i, \qquad b(a_i) \equiv z_i \pmod 1.$$

Then, at an arbitrary element a of A, cf. (2:4), the value of the characters b will be given by the expression

$$b(a) \equiv \Sigma \xi_i \eta_i + \Sigma x_i z_i \pmod 1, \tag{3:2}$$

therefore b will be fully determined. Now, by Condition (ii), it follows immediately that every character b must vanish, modulo 1, at the zero element of A. Therefore, in virtue of Relations (2:2), every set of numbers (3:1) determining a character b will have to be a solution of the congruences

$$\Sigma p_{is}\eta_s + \Sigma q_{is}z_s \equiv 0 \pmod 1. \tag{3:3}$$

Conversely, every solution (3:1) of the congruences (3:3) will determine a character. For the functions b determined by (3:2) will then have a unique value, modulo 1, at each element a of A, regardless of how we express the element a in terms of the base elements of A. Moreover, the function b will obviously fulfil the two requirements imposed at the beginning of the section and will therefore be a character.

It will be more convenient to replace the congruences (3:3) by an equivalent set of equations

$$\Sigma p_{is}\eta_s + \Sigma q_{is}z_s = y_i, \tag{3:4}$$

where the new variables y_i are allowed to range over all *integer* values. These equations will be solvable for the variables z_i in terms of the variables y_s and η_s. For we know that in the left-hand member of the *ith* equations the coefficient q_{ii} of the last term is always different from zero. Therefore we can express the variable z_i as a finite linear combination of the variables z_s preceding z_i, of the variables η_s, and of the variable y_i. It follows immediately, by transfinite induction, that we can express the variable z_i as a *finite* linear combination

$$z_i = \Sigma m_{is}\eta_s + \Sigma n_{is}y_s \tag{3:5}$$

of the variables η_s and y_s alone. An easy computation shows that if M is the matrix of coefficients m_{is} and N the matrix of coefficients n_{is}, then we must have $N = Q^{-1}$ and $M = -Q^{-1}P$, but the exact form of the matrices M and N need not concern us here.

Relations (3:5) tell us that to *every* choice of the numbers

$$(\eta_1, \eta_2, \cdots ; \quad y_1, y_2, \cdots) \tag{3:6}$$

where the η_i's are arbitrary *real* numbers and the y_i's arbitrary *integers*, there corresponds a set of numbers (3:1) satisfying the congruences (3:3) and, therefore, a character b. Conversely, to every character b there corresponds at least one set of numbers (3:1) satisfying the congruences (3:3) and, therefore, by (3:4), at least one set of numbers (3:6). We can thus form what may, perhaps, be called a "pseudo basis" for the characters b consisting of a set of characters

$$\beta_1, \beta_2, \cdots ; \quad b_1, b_2, \cdots , \tag{3:7}$$

such that β_i is the character determined by setting the number η_i equal to unity and all the other numbers in (3:6) equal to zero, and such that b_i is the character determined by setting the number y_i equal to unity and all the others equal to

zero. In terms of the elements (3:7) an arbitrary character b corresponding to an arbitrary choice of the numbers (3:6) will be expressible as an *infinite* linear form

$$b = \Sigma \eta_i \beta_i + \Sigma y_i b_i \tag{3:8}$$

with *real* coefficients η_i and *integer* coefficients y_i. In order that the representation (3:7) have a significance we must show that to derive the sum or difference of two characters b_1 and b_2, we have only to add or subtract corresponding coefficients in the representative forms of b_1 and b_2. This is obvious, however; for the value of a character b at an element a of A is a linear form (3:2) in the coordinates (3:6) of b involving, for each particular element a, only a *finite* number of coordinates of b.

To describe the character group B completely we shall still have to show under what conditions two linear forms (3:8) determine the same character b. This problem reduces, of course, to the one of finding all linear forms representing the zero element of B. Now, obviously, the form (3:8) will represent the zero element if and only if its coefficients are such as to make all the numbers (3:1) integers (so that they reduce to zero, modulo 1). We can therefore construct a "fundamental set" consisting of all the forms (3:8) determined by setting one of the numbers (3:1) equal to unity and all of the others equal to zero. Thus, by making η's unity and the other numbers zero we obtain the form

$$M_s = \beta_s + \Sigma p_{is} b_i = 0 \tag{3:9}$$

and by making z_s unity and the other numbers zero, the form

$$N_s = \Sigma q_{is} b_i = 0. \tag{3:10}$$

The form corresponding to an arbitrary choice of the integers (3:1) will be the *infinite* form

$$\sum_s \eta_s M_s + \sum_s z_s N_s = \sum_s \eta_s \beta_s + \sum_i \sum_s (\eta_s p_{is} + z_s q_{is})\, b_i = 0, \tag{3:11}$$

where the coefficients η_s and z_s are arbitrary integers. The coefficient $\sum_s (\eta_s p_{is} + z_s q_{is})$ of b_i in (3:11) will, of course, be a finite sum, so that the form (3:11) will always have a perfectly definite meaning. If we regard the coefficients p_{is} in (3:9) as the elements of a matrix P', then, clearly, P' will be the transposed of the matrix P defined in §2. For, in the present case, we are summing over the first index i and not, as before, over the second index s. By a similar argument, the matrix Q' of the coefficients q_{is} in (3:10) will be the transposed of the matrix Q.

It would be incorrect to argue from Relations (3:9) that the base elements β_i were redundant. These relations enable us to eliminate arbitrary *integer* multiples of the β_i's but not arbitrary *real* multiples.

4. The original group A associated with the character group B can be obtained from B by dualizing the process of §3. Let

$$(v_1, v_2, \cdots ; \quad w_1, w_2, \cdots) \tag{4:1}$$

be any set of *real* numbers such that at most a finite number of them are different from zero. Then the numbers (4:1) determine a (single or multiple valued) function c of an element b of B such that the values of c in terms of the coordinates of b are

$$c(b) \equiv \Sigma v_i \eta_i + \Sigma w_i y_i \pmod 1. \tag{4:2}$$

The function $c(b)$ will be *single valued* provided its value does not depend on the particular choice of the coordinates of b: that is to say, provided the numbers (4:1) satisfy the congruences

$$\begin{aligned} v_s + \Sigma p_{is} w_s &\equiv 0 \\ \Sigma q_{is} w_s &\equiv 0 \end{aligned} \pmod 1. \tag{4:3}$$

Conversely, if the numbers (4:1) satisfy the congruences (4:3) then the functions $c(b)$ will be single valued. Moreover, $c(b)$ will have the property that if b and b' are any two group elements of B, then we shall have

$$c(b + b') \equiv c(b) + c(b') \pmod 1. \tag{4:4}$$

In other words, the function c will be a character of B. We shall denote the group formed by the characters (4:2) by C. Of course, we do not assert that C is the *complete* character group of B.

Now, the congruences (4:3) may evidently be replaced by an equivalent set of equations

$$\begin{aligned} v_s + \sum_i p_{is} w_i &= \xi_s \\ \sum_i q_{is} w_i &= x_s \end{aligned} \tag{4:5}$$

where the new variables ξ_s and x_s are arbitrary *integers* such that at most a *finite* number of them are different from zero. For corresponding to each allowable choice of the numbers (4:1) the total number of non-vanishing terms in the combined left-hand members of (4:5) will be finite; therefore the same will be true about the number of non-vanishing terms in the right-hand members. Equations (4:5) may be solved for the w_s's and v_s's. By inspection, we see that the solution will be

$$\begin{aligned} w_j &= \sum_s n_{sj} x_s \\ v_j &= \xi_j + \sum_s m_{sj} x_s, \end{aligned} \tag{4:6}$$

where the coefficients m_{sj} and n_{sj} have the same meaning as in Relations (3:5). Thus, to every determination of the *integers*

$$(\xi_1, \xi_2, \cdots ; x_1, x_2, \cdots) \tag{4:7}$$

such that at most a finite number of them are different from zero, there will correspond a character c. Conversely, to every character c of the form (4:2) there will correspond one or more sets of integers (4:7). We can thus form a basis of C of the ordinary sort (§2), consisting of a set of characters of the form

$$\gamma_1, \gamma_2, \cdots ; c_1, c_2, \cdots , \tag{4:8}$$

where γ_i is the character determined by setting ξ_i equal to unity and the other integers equal to zero and where c_i is the character determined by setting x_i equal to unity and the other integers equal to zero. A general character c will then be a finite sum of the form

$$c = \Sigma \xi_s \gamma_s + \Sigma x_s c_s \tag{4:9}$$

The character c will vanish if and only if the coefficients ξ_s and x_s are so chosen as to make all the numbers v_s in (4:1) vanish, and as to make all the numbers w_s vanish modulo 1. We emphasize the fact that the numbers v_s must actually *vanish*, and not merely vanish modulo 1, because the coefficients η_i in (4:2) are arbitrary real numbers and not arbitrary integers. We can therefore obtain a fundamental set for the group C by constructing all the forms (4:9) obtainable by setting one of the integers w_i equal to unity and all the other integers in (4:1) equal to zero. In this way we shall be led to the relations

$$\sum_s p_{is} \gamma_s + \sum_s q_{is} c_s = 0. \tag{4:10}$$

By comparing (4:10) with (2:2) we see, at once, that the group C is simply isomorphic with the group A from which we started.

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SUR LA TOPOLOGIE DE CERTAINS ESPACES HOMOGÈNES

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Préface. Les propriétés d'un espace homogène dans lequel opère un groupe transitif de Lie expriment simplement les propriétés de ce groupe. Il serait intéressant de savoir les relations entre la topologie d'un tel espace et les propriétés de son groupe de structure. Nos connaissances à ce sujet sont encore très incomplètes. Cependant, dans ses recherches concernant les groupes simples et les espaces homogènes symétriques, M. E. Cartan est arrivé à des résultats remarquables qui font connaître certaines de ces relations. Un des résultats les plus surprenants ramène la détermination des nombres de Betti d'un espace riemannien symétrique clos à un problème purement algébrique concernant le groupe linéaire d'isotropie de cet espace. Il existe une classe d'espaces riemanniens symétriques clos qui sont réalisés par des variétés algébriques complexes. Dans ce mémoire nous nous proposons d'étudier les propriétés topologiques des espaces de cette classe. La méthode indiquée par M. E. Cartan pour calculer les nombres de Betti revient, pour ces espaces, à décomposer certains groupes linéaires en groupes irréductibles. Ce sera l'une des méthodes employées. Comme les espaces considérés ont des définitions géométriques simples, nous avons cherché une deuxième méthode, qui utilise des procédés habituels à la topologie, à savoir les déformations et les subdivisions en cellules. Par cette deuxième méthode on peut déterminer d'une façon élémentaire les bases d'homologie et les groupes de Poincaré. Nous étudierons ainsi tous les espaces de la classe considérée et nous appliquerons la deuxième méthode à certaines autres familles de variétés algébriques. Nous démontrerons par la topologie certaines formules de géométrie énumérative, en particulier une formule importante due à H. Schubert.

Pour faciliter la lecture de ce mémoire, nous rappelons dans la première section quelques notions et quelques résultats qui jouent un rôle fondamental dans les travaux déjà mentionnés de M. E. Cartan. En ce qui concerne la topologie, nous renvoyons à un ouvrage de M. S. Lefschetz: "*Topology*" (voir l'index bibliographique). Au sujet des propriétés classiques des variétés étudiées, on trouvera des renseignements très complets dans un article de C. Segre: *Mehrdimensionale Räume.*

Qu'il me soit permis d'exprimer ici ma profonde reconnaissance à M. Elie Cartan et à M. Salomon Lefschetz qui ont bien voulu montrer de l'intérêt pour mon travail. Leurs critiques et leurs suggestions m'ont grandement aidé au cours de mes recherches.

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I. Généralités sur les espaces homogènes[1]

1. **Définitions et propriétés générales.** Un *espace homogène* du type le plus général est un espace topologique dont le groupe des *homéomorphies* est

[1] Voir E. Cartan, *a* et *b*. Pour les titres complets des articles et des ouvrages cités voir l'index bibliographique.

transitif. Dans la suite nous réservons le nom d'espace homogène à une variété topologique à n dimensions qui est transformée transitivement par un groupe fini et continu G. Ce groupe sera appelé le *groupe de structure* de l'espace et nous supposerons que c'est un groupe fini et continu de Lie.

Soit E un tel espace homogène, G son groupe de structure et g le plus grand sous-groupe de G qui laisse invariant un point O de l'espace. g est un sous-groupe fermé dans G. En outre g ne contient pas de sous-groupe invariant dans G, si aucune transformation de G ne laisse fixes tous les points de E. G est supposé connexe. Le groupe g s'appelle *groupe d'isotropie* ou groupe des rotations autour de O.

A un point M arbitraire de E on peut faire correspondre l'ensemble Sg de toutes les transformations de G amenant O en M. On les obtient en effectuant successivement une transformation arbitraire de g et une transformation particulière S qui amène O en M. Réciproquement étant donné un groupe G, soit g un sous-groupe fermé dans G. L'ensemble de G et de g peut servir à définir un espace homogène E dont les points correspondent aux ensembles de transformations Sg de G. Il y aura une correspondance ponctuelle entre les points de E et les points de la variété du groupe G. Un point de G a pour image un point unique M de E, mais un point M de E correspond à un ensemble de points Sg de G. Un voisinage du point M dans E se compose des images des ensembles sSg, où s est un élément arbitraire d'un voisinage V_0 de l'élément unité dans G. Ainsi un voisinage du point O se compose des images des ensembles sg. Supposons que les paramètres dans une région autour de l'élément unité de G soient des paramètres canoniques. Chaque transformation dans cette région est alors représentée d'une manière et d'une seule par le symbole $e_1X_1 + e_2X_2 + \cdots + e_nX_n + e_{n+1}X_{n+1} + \cdots + e_rX_r$. Le sous-groupe g, qui est fermé dans G, est un groupe de Lie et nous supposerons que sa partie connexe contenant l'élément unité est engendrée par $X_{n+1}, \cdots, X_{r-1}, X_r$. Si V_0 est suffisamment petit, tout élément s de V_0 est le produit d'une transformation bien déterminée de g par une transformation bien déterminée t de V_0 représentée par $e_1X_1 + e_2X_2 + \cdots + e_nX_n$. Cela veut dire que la variété sg coupe l'hyperplan $e_{n+1} = e_{n+2} = \cdots = e_r = 0$ en un seul point t à l'intérieur de V_0. Le point P qui représente dans E l'ensemble sg est en correspondance biunivoque avec le point t de paramètres $(e_1, e_2, \cdots, e_n)$. Tout autre point de G correspondant au point P est représenté d'une manière et d'une seule par un produit tg. Donc la région de G qui a pour image sur E un voisinage V_0' suffisamment petit de O est le produit topologique de V_0' et de g.

Une transformation T de G fait correspondre au point M, image de Sg, le point M', image de TSg. En particulier S^{-1} transforme le voisinage V_0Sg du point M en un voisinage $S^{-1}V_0Sg$ du point O. Les points de ce dernier voisinage sont encore en correspondance biunivoque avec des points t de paramètres $(e_1, e_2, \cdots, e_n)$. En revenant au point M par la transformation S, on voit qu'un voisinage V_0' de M a pour image dans G le produit topologique de V_0' et de g. Cette propriété a lieu pour un voisinage suffisamment petit d'un point quelconque de E; mais, en général, la variété de G n'est pas globalement

le produit topologique de E et de g. E est un espace de décomposition pour la variété G. Les propriétés d'un espace homogène ne traduisent ou ne révèlent que des propriétés de son groupe de structure. Cette remarque s'applique en particulier aux propriétés *topologiques*. L'existence même de la décomposition de la variété G qui fournit l'espace homogène E est une propriété topologique de G, qui permettrait peut-être d'arriver encore à de nouveaux résultats.

2. **Groupe de Poincaré d'un espace homogène.** Le groupe de Poincaré de l'espace homogène E dépend d'une façon assez simple des propriétés de l'ensemble de G et de g.

Toute courbe dans G a pour image une courbe dans l'espace homogène E. Tout arc OM dans E peut être recouvert par un nombre fini de voisinages V_0 dont chacun a pour image dans G une région qui est le produit topologique de ce voisinage avec g. On obtient alors sans peine un arc continu IS dans G qui a pour image l'arc OM. On peut aussi trouver dans G deux arcs voisins ayant pour images dans E deux arcs voisins donnés. Si l'on déforme l'arc OM d'une façon continue en laissant les points O et M fixes, on peut définir une déformation correspondante de l'arc IS, I restant fixe et S se déplaçant en général sur l'ensemble Sg. Etant donnée dans E une courbe fermée orientée C partant de O et y revenant, il lui correspond dans G une courbe orientée allant de l'élément unité I à un point J de g.

Le sous-groupe g peut se composer de plusieurs parties connexes $g_0, g_1, \cdots, g_k$. La condition nécessaire et suffisante pour que la courbe C soit réductible à O par déformation continue est qu'il lui corresponde dans G une courbe IJ qui soit déformable en une courbe située sur g_0, le point I étant fixe et J pouvant se déplacer d'une façon continue sur g. Le point J se trouvera alors nécessairement sur g_0. On peut alors définir dans la variété de G un groupe Γ isomorphe au groupe de Poincaré de l'espace E. Une courbe allant de I à un point J_α de la variété g_α représente un élément du groupe. Soient deux courbes IJ_α et IJ_β. Si S est le point général de la courbe IJ_β, le point SJ_α décrit une courbe $J_\alpha J_\gamma$ allant du point J_α au point $J_\gamma = J_\beta J_\alpha$. La courbe composée de IJ_α et $J_\alpha J_\gamma$ représentera le produit des deux éléments de Γ correspondant à IJ_α et IJ_β. L'élément unité du groupe sera représenté par toutes les courbes IJ_0 qui sont déformables en des courbes situées sur g_0, en supposant I fixe et J_0 mobile sur g_0. Le groupe Γ a pour sous-groupe invariant le groupe Γ_0 représenté par les courbes IJ_0. Le groupe quotient Γ/Γ_0 est isomorphe au groupe quotient g/g_0. En particulier, *si g est connexe le groupe Γ se réduit à Γ_0. Si le groupe G est simplement connexe, Γ se réduit à g/g_0.*

Si G n'est pas simplement connexe, on peut considérer la variété simplement connexe de recouvrement de G. C'est la variété d'un groupe $\bar{G}$. g_0 a pour image dans $\bar{G}$ une variété $\bar{g}_0$ formant un sous-groupe de $\bar{G}$. $\bar{g}_0$ n'est pas nécessairement connexe. Si $\bar{g}_0'$ est la partie connexe de $\bar{g}_0$ contenant l'élément unité, le groupe discontinu $\bar{g}_0/\bar{g}_0'$ est isomorphe au groupe Γ_0. Si $\bar{g}$ est le plus grand sous-groupe de $\bar{G}$ recouvrant g, le groupe Γ est isomorphe au groupe quotient $\bar{g}/\bar{g}_0'$.

3. **Espaces homogènes orientables et non orientables.** Considérons un point P infiniment voisin de O. Il définit un vecteur infiniment petit $\overrightarrow{OP}$ ou une direction issue de O. Le point P se déduit de O par une transformation infinitésimale $e_1X_1 + \cdots + e_nX_n$ bien déterminée et les quantités $e_1, e_2, \cdots, e_n$ peuvent être considérées comme les composantes du vecteur $\overrightarrow{OP}$. Etant donnés n vecteurs infiniment petits $\overrightarrow{OP}_1, \overrightarrow{OP}_2, \cdots, \overrightarrow{OP}_n$, nous dirons que le simplexe $OP_1P_2 \cdots P_n$ est de sens positif lorsque le déterminant $|e_i^{(j)}|$ des composantes de ces vecteurs est positif. Si le déterminant a une valeur négative, le simplexe est dit de sens négatif. Etant donnée une courbe allant de O à M, on peut la considérer comme image d'une courbe dans la variété G allant de l'élément unité I à un élément S. La courbe IS définit un déplacement continu le long de la courbe OM amenant le simplexe $OP_1P_2 \cdots P_n$ en un simplexe $MQ_1Q_2 \cdots Q_n$. Soit en particulier une courbe fermée (C) passant par O. On peut lui faire correspondre dans G une courbe allant de I à un élément R qui appartient à g. Le déplacement le long de la courbe (C) amène le simplexe $OP_1P_2 \cdots P_n$ en un simplexe $OQ_1Q_2 \cdots Q_n$. Un point P correspondant à une transformation infinitésimale s est amené au point Q correspondant à la transformation Rs ou $(RsR^{-1})R$. Le point Q est donc défini par la transformation infinitésimale RsR^{-1} qui se déduit de s par une opération du groupe adjoint linéaire de G. Lorsque R varie d'une façon quelconque dans g, cette opération engendre un sous-groupe du groupe adjoint linéaire. Comme elle laisse invariantes les transformations infinitésimales de g, les variables $e_1, \cdots, e_n$ sont transformées entre elles suivant les transformations d'un groupe linéaire γ. Ce groupe indique simplement comment le groupe g transforme entre eux les vecteurs infiniment petits d'origine O. Nous l'appellerons *groupe linéaire d'isotropie*. Si le déterminant de l'opération de γ correspondant à R est positif, le simplexe $OQ_1Q_2 \cdots Q_n$ a le même sens que le simplexe primitif. Si ce déterminant est négatif, le déplacement le long de la courbe fermée (C) ramène le simplexe au point O avec un sens opposé.

L'espace E sera orientable lorsque le déterminant de γ est toujours positif. Il sera non orientable lorsque ce déterminant n'a pas son signe constant. En particulier si g est connexe, le déterminant de γ ne peut pas changer de signe et l'espace E est orientable. Ainsi la variété d'un groupe est toujours orientable.

Le groupe γ est en relation avec d'autres propriétés de l'espace E. Rappelons que l'existence dans E d'une métrique riemannienne invariante par G revient à l'existence d'une forme quadratique définie positive en $e_1, e_2, \cdots, e_n$ invariante par le groups γ.

4. **Les invariants intégraux d'un espace homogène à groupe fondamental clos.**[2] Un invariant intégral de degré p de l'espace homogène E est une intégrale $\int\Omega$ de degré p dont l'élément Ω est invariant par G. Ω est une forme extérieure $\Sigma A_{i_1i_2\cdots i_p}[dx_{i_1}\, dx_{i_2} \cdots dx_{i_p}]$ dont les coefficients $A_{i_1i_2\cdots i_p}$ sont des fonctions

[2] Voir E. Cartan, *b*.

uniformes des coordonnées $x_1, x_2, \cdots, x_n$ d'un point M dans l'espace E. Les produits extérieurs $[dx_{i_1} dx_{i_2} \cdots dx_{i_p}]$ sont les composantes d'un p-vecteur formé de p vecteurs infiniment petits d'origine M.

La forme Ω sera complètement déterminée par sa valeur Ω^0 au point O, soit $\Omega^0 = \Sigma A^0_{i_1 i_2 \cdots i_p} [dx_{i_1} dx_{i_2} \cdots dx_{i_p}]$. On peut considérer comme composantes d'un vecteur infiniment petit $\overrightarrow{OP}$ soit les différentielles correspondantes $dx_1, \cdots, dx_n$, soit les paramètres $\omega_1, \omega_2, \cdots, \omega_n$ de la transformation infinitésimale $\omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_n X_n$ qui amène O en P. Les ω_i sont des formes linéaires indépendantes par rapport aux dx_i. Donc Ω^0 peut se mettre sous la forme $\Sigma B_{i_1 i_2 \cdots i_p} [\omega_{i_1} \omega_{i_2} \cdots \omega_{i_p}]$. Une transformation R du groupe des rotations g amène le vecteur $\overrightarrow{OP}$ de composantes ω_i en un vecteur $\overrightarrow{OP'}$ dont les composantes $\bar{\omega}_i$ se déduisent des ω_i par une transformation du groupe linéaire γ. Le fait que Ω^0 est invariant par g se traduit par l'égalité:

$$\Sigma B_{i_1 i_2 \cdots i_p} [\bar{\omega}_{i_1} \bar{\omega}_{i_2} \cdots \bar{\omega}_{i_p}] = \Sigma B_{i_1 i_2 \cdots i_p} [\omega_{i_1} \omega_{i_2} \cdots \omega_{i_p}]$$

c'est-à-dire la forme $\Sigma B_{i_1 i_2 \cdots i_p} [\omega_{i_1} \omega_{i_2} \cdots \omega_{i_p}]$ est une forme extérieure à coefficients constants invariante par le groupe linéaire γ.

Réciproquement toute forme extérieure $\Sigma B_{i_1 i_2 \cdots i_p} [\omega_{i_1} \omega_{i_2} \cdots \omega_{i_p}]$ à coefficients constants invariante par γ définit un invariant intégrale Ω de l'espace E.

Donc *la recherche des invariants intégraux de l'espace homogène E revient à la recherche des formes extérieures $\Sigma B_{i_1 i_2 \cdots i_p} [\omega_{i_1} \omega_{i_2} \cdots \omega_{i_p}]$ invariantes par le groupe linéaire γ.*

Si le groupe de structure G de l'espace E est clos on a deux théorèmes importants démontrés par M. E. Cartan:

a) *Toute intégrale de différentielle exacte est équivalente à un invariant intégral.*

b) *Tout invariant intégral équivalent à zéro est la dérivée extérieure d'un invariant intégral dont le degré est moindre d'une unité.*

Les intégrales que l'on considère sont des intégrales multiples $\int \Omega$ régulières[3] dans tout l'espace. Rappelons qu'une telle intégrale $\int \Omega$ est dite intégrale de différentielle exacte, si la dérivée extérieure Ω' de Ω est identiquement nulle. L'intégrale $\int \Omega$ est dite équivalente à zéro, si la forme Ω est la dérivée extérieure d'une forme Π dont le degré est moindre d'une unité. Plusieurs intégrales de différentielles exactes sont dites linéairement indépendantes, lorsqu'aucune combinaison linéaire de ces intégrales n'est équivalente à zéro.

Il résulte des théorèmes (*a*) et (*b*) que, si l'espace E admet en tout n_p formes invariantes de degré p, linéairement indépendantes au sens algébrique ordinaire, et si ν_p de ces formes sont à dérivée extérieure nulle, *le nombre d'intégrales de différentielles exactes linéairement indépendantes de degré p est égal à $\nu_p + \nu_{p-1} - n_{p-1}$.*

Un espace homogène E dont le groupe de structure est un groupe de Lie forme une variété analytique sans singularités. Il en existe par suite une subdivision polyédrale définissant une multiplicité (en anglais "manifold") du point

[3] Voir G. de Rham, *a*.

de vue de la topologie combinatoire.[4] G étant clos, l'espace E forme une variété analytique close à n dimensions dont une subdivision polyédrale fournit un complexe analytique fini. Cette variété rentre dans la catégorie des variétés closes au sujet desquelles M. G. de Rham a démontré le résultat fondamental suivant:[5]

Le nombre d'intégrales de différentielles exactes de degré p linéairement indépendantes est égal au nombre de Betti relatif à la dimension p.

La considération des invariants intégraux fournit ainsi une méthode pour déterminer les nombres de Betti d'un espace homogène à groupe fondamental clos. Soit R_p le nombre de Betti relatif à la dimension p. On a: $R_p = \nu_p + \nu_{p-1} - n_{p-1}$. Cette méthode se simplifie dans le cas des espaces homogènes symétriques.

5. **Les espaces homogènes symétriques.**[6] Un espace homogène E qui a G pour groupe de structure et g pour groupe d'isotropie au point O est dit *symétrique* s'il existe une transformation ponctuelle σ_0 de cet espace ayant les propriété suivantes:

1. σ_0 est involutive.
2. O est un point invariant isolé pour σ_0.
3. σ_0 est une automorphie de G.
4. La partie connexe de g contenant la transformation identique est le plus grand sous-groupe continu de G dont toutes les transformations sont invariantes par σ_0. L'existence d'une "*symétrie*" σ_0 par rapport au point O entraîne l'existence d'une symétrie analogue par rapport à un point quelconque de l'espace et toutes ces symétries sont transformées entre elles par le groupe G.

Les résultats rappelés au §4 se simplifient ici en vertu du théorème suivant:

Tout invariant intégral d'un espace homogène symétrique est une intégrale de différentielle exacte.

Donc si l'espace homogène symétrique E est *clos, le nombre d'intégrales de différentielles exactes linéairement indépendantes de degré p, ou encore le nombre de Betti R_p, est égal au nombre d'invariants intégraux de degré p linéairement indépendants dans le sens algébrique ordinaire.* On connaîtra les nombres de Betti si l'on sait trouver le nombre d'invariants linéaires du groupe linéaire γ_p qui indique comment les formes extérieures $[e_{i_1} e_{i_2} \cdots e_{i_p}]$ sont transformées par le groupe linéaire d'isotropie γ.

γ étant un groupe clos, il existe une forme quadratique définie positive invariante par γ et par suite une métrique riemannienne à ds^2 défini positif invariante par G.

La détermination des espaces riemanniens symétriques se ramène à la détermination de ceux que sont dits *irréductibles.* Pour un espace irréductible, le

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[4] Voir au sujet de cette subdivision: S. Lefschetz, *a*, p. 364–366. S. Lefschetz and J. H. C. Whitehead, *a*. B. O. Koopman and A. B. Brown, *a*.

[5] G. de Rham, *a*, chap. III, p. 73.

[6] Voir E. Cartan, *a*, *b* et *c*.

groupe γ est irréductible dans le domaine réel des variables $e_1, e_2, \cdots, e_n$. Laissons de côté les espaces riemanniens symétriques ouverts qui sont tous homéomorphes à l'espace euclidien. Les espaces symétriques irréductibles clos se répartissent en deux catégories. La première catégorie comprend les *espaces des groupes simples clos.*[7] La deuxième catégorie[8] comprend les *espaces dont le groupe de structure G est un groupe simple clos, le groupe d'isotropie g,* ou au moins sa partie connexe contenant l'élément unité, *étant le plus grand sous-groupe continu de G dont toutes les transformations sont invariantes par une automorphie involutive de G.*

Pour les espaces de la deuxième catégorie, deux cas peuvent se présenter. Dans le premier cas, le groupe linéaire d'isotropie est un groupe linéaire simple ou semi-simple et *irréductible* même dans le domaine *complexe* des variables $e_1, e_2, \cdots, e_n$. Dans le deuxième cas, le groupe linéaire γ, qui est toujours irréductible dans le domaine réel des variables $e_1, e_2, \cdots, e_n$, est *réductible* dans le domaine *complexe* de ces variables. Il transforme entre elles $n/2$ variables $u_1, u_2, \cdots, u_{n/2}$, combinaisons linéaires à coefficients complexes des $e_1, e_2, \cdots, e_n$, ainsi que les $n/2$ variables $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_{n/2}$, combinaisons linéaires de $e_1, e_2, \cdots, e_n$ à coefficients complexes conjugués des précédents. Le groupe γ, opérant sur ces nouvelles variables, laisse invariante la forme d'Hermite

$$u_1\bar{u}_1 + u_2\bar{u}_2 + \cdots + u_{n/2}\bar{u}_{n/2}$$

et contient toujours le groupe à un paramètre $u_k' = e^{i\theta}u_k$. γ sera le produit direct de ce groupe à un paramètre et d'un groupe simple ou semi-simple γ'. Les espaces correspondant à ce deuxième cas forment une *classe d'espaces riemanniens symétriques clos qu'on peut considérer comme analytiques complexes.* Considérons particulièrement les espaces de cette dernière classe.

Pour déterminer les nombres de Betti d'un tel espace, il faut déterminer le nombre d'invariants linéaires du groupe γ_p qui indique comment γ transforme les formes extérieures $[e_{i_1} e_{i_2} \cdots e_{i_p}]$, ou $[u_{i_1} \cdots u_{i_\alpha} \bar{u}_{i_{\alpha+1}} \cdots \bar{u}_{i_p}]$ avec les variables u_i et $\bar{u}_i$. Une combinaison linéaire de ces formes n'est invariante par la transformation

$$u_k' = e^{i\theta}u_k$$
$$\bar{u}_k' = e^{-i\theta}\bar{u}_k$$

que si elle se compose de formes ayant autant de variables u_k que de variables $\bar{u}_k$. En particulier il n'existe pas de forme extérieure invariante de degré impair. Donc les nombres de Betti relatifs aux dimensions impaires sont nuls.

Considérons alors les formes extérieures de degré $2s$, combinaisons linéaires des formes $[u_{i_1} \cdots u_{i_s} \bar{u}_{j_1} \cdots \bar{u}_{j_s}]$. Elles sont transformées par γ' suivant un certain groupe linéaire γ'_{2s}. Le nombre de formes de degré $2s$ invariantes par γ, c'est-à-dire le nombre d'invariants linéaires du groupe γ'_{2s}, est égal à la valeur

[7] Voir E. Cartan, *d.*
[8] Voir E. Cartan, *e.*

moyenne du caractère de γ'_{2s}.[9] On voit que le caractère de γ'_{2s} est le produit des caractères des groupes linéaires γ''_s et $\bar{\gamma}''_s$ qui indiquent comment γ' transforme les formes $[u_{i_1} u_{i_2} \cdots u_{i_s}]$ et $[\bar{u}_{i_1} \bar{u}_{i_2} \cdots \bar{u}_{i_s}]$. Donc le caractère de γ'_{2s} est égal au carré du module du caractère de γ''_s. Pour en trouver la valeur moyenne, on peut décomposer le groupe linéaire γ''_s en groupes linéaires irréductibles. Si l'on obtient k groupes linéaires irréductibles non équivalents entre eux, ces groupes figurant respectivement ν_1 fois, ν_2 fois, $\cdots$, ν_k fois dans la décomposition, la valeur moyenne du carré du module du caractère de γ''_s est égale à

$$\nu_1^2 + \nu_2^2 + \cdots + \nu_k^2.$$

C'est aussi le nombre de Betti R_{2s}.

En particulier γ''_1 est irréductible et par suite les nombres de Betti R_2 et R_{n-2} sont égaux à l'unité.

Abstraction faite de deux espaces exceptionnels correspondant à un groupe simple de type exceptionnel, cette dernière classe d'espaces symétriques clos comprend les espaces suivants:

L'espace projectif complexe.

L'espace des variétés planes à p dimensions de l'espace projectif complexe à n dimensions.

La quadrique complexe non dégénérée.

L'espace des variétés planes à p dimensions génératrices de la quadrique complexe à 2p dimensions.

L'espace des variétés planes à p dimensions qui appartiennent à un complexe linéaire non dégénéré de l'espace projectif complexe à $2p + 1$ dimensions.

L'étude des espaces de cette classe sera l'objet principal de ce mémoire. Nous appliquerons la méthode que nous venons de rappeler et qui conduit à des problèmes intéressants concernant certains groupes linéaires.[10] Comme tous les espaces de cette classe ont une définition géométrique très simple, il sera intéressant d'étudier leur topologie aussi par une méthode plus directe qui donne des renseignements plus précis et qui s'applique à un grand nombre d'autres variétés algébriques.

II. Les variétés de Grassmann considérées comme des espaces homogènes symétriques

6. **Le groupe de structure et le groupe d'isotropie.** L'ensemble des variétés planes à k dimensions d'un espace projectif complexe à n dimensions définit une variété algébrique V appelée *variété de Grassmann*. En définissant chaque variété plane à k dimensions par l'ensemble de ses $\binom{n+1}{k+1}$ coordonnées plückeriennes homogènes $p_{i_0 i_1 \cdots i_k}$, on obtient une représentation biunivoque de la

[9] Voir pour la théorie des caractères: H. Weyl, *a*.

[10] Au sujet de l'application de cette méthode à l'espace projectif complexe, à l'espace des droites de l'espace projectif à 3 dimensions et à la quadrique complexe, voir E. Cartan, *b* et *g*.

variété V par une variété algébrique sans singularités de l'espace projectif à $\binom{n+1}{k+1} - 1$ dimensions. (Nous désignons par $\binom{a}{b}$ le nombre de combinaisons b à b de a éléments.) La variété V est connexe, car elle est transformée transitivement par le groupe projectif complexe, qui est connexe.

La variété V peut être considérée comme un espace homogène symétrique clos. Considérons en effet le groupe hermitien elliptique, c'est-à-dire le groupe projectif G

$$x_i' = \Sigma\, a_{ij}\, x_j, \qquad (i, j, = 0, 1, \cdots, n) \tag{1}$$

qui laisse invariante la forme d'Hermite $x_0\,\bar{x}_0 + x_1\,\bar{x}_1 + \cdots + x_n\,\bar{x}_n$. Si A est la matrice de la transformation (1) et $\bar{A}^*$ la matrice imaginaire conjuguée transposée, on a: $A\,\bar{A}^* = 1$. Nous pouvons supposer $|A| = 1$. Les transformations infinitésimales sont:

$$\delta x_i = \Sigma\, \alpha_{ij}\, x_j, \qquad \alpha_{ij} + \bar{\alpha}_{ji} = 0.$$

Nous désignerons toujours par $[k]$ une variété plane à k dimensions. Soit la variété plane $[k]_0$ définie par les équations $x_{k+1} = x_{k+2} = \cdots = x_n = 0$. En appelant $y_0, y_1, \cdots, y_q$ les $n - k$ dernières variables $x_{k+1}, x_{k+2}, \cdots, x_n$, le plus grand sous-groupe g de G qui laisse invariante la variété plane $[k]_0$ a pour équations:

$$\text{(g)} \qquad \begin{aligned} x_{j'}' &= \Sigma\, a_{j'j}\, x_j \qquad (j, j' = 0, 1, \cdots, k) \\ y_{i'}' &= \Sigma\, b_{i'i}\, y_i \qquad (i, i' = 0, 1, \cdots, q) \end{aligned}$$

C'est aussi le plus grand sous-groupe continu de G qui soit invariant par l'involution:

$$x_0' = x_0, x_1' = x_1, \cdots, x_k' = x_k, y_0' = -y_0, y_1' = -y_1, \cdots, y_q' = -y_q.$$

Cette involution est une automorphie involutive de G. L'ensemble de G et de g définit donc un espace homogène symétrique clos.

Chaque transformation infinitésimale de G se décompose d'une manière et d'une seule en une transformation infinitésimale de g et une transformation infinitésimale de la forme:

$$\begin{aligned} \delta x_j &= -\Sigma\, \bar{\omega}_{ij}\, y_i \qquad (i = 0, 1, \cdots, q) \\ \delta y_i &= \Sigma\, \omega_{ij}\, x_j \qquad (j = 0, 1, \cdots, k) \end{aligned}$$

Celle-ci transforme la variété $[k]_0$ d'équations $y_0 = y_1 = \cdots = y_q = 0$ en une variété $[k]$ d'équations:

$$y_i = \omega_{ij}\, x_j. \tag{2}$$

Les $(k + 1)\,(n - k)$ paramètres complexes ω_{ij} peuvent être considérés comme les coordonnées complexes de cette variété plane. Les équations de toute variété plane $[k]$ voisine de $[k]_0$ peuvent se mettre sous la forme (2). Le groupe G est donc transitif dans la variété de Grassmann V au voisinage de l'élément $[k]_0$. Les transformés de $[k]$ par le groupe clos G engendrent une variété close

V', qui devra être une région de V n'ayant pas de frontière dans V. Par suite V' est confondu avec V. *La variété de Grassmann V peut donc être considérée comme l'espace riemannien symétrique clos défini par G et g.*

Sous forme symbolique, les équations de g s'écrivent:

$$(x') = (a)\,(x)$$

$$(y') = (b)\,(y)$$

avec les conditions: $(a)\ (\bar{a})^* = 1$, $(b)\ (\bar{b})^* = 1, \left|a\right| \cdot \left|b\right| = 1$. Soit (ω) la matrice des paramètres ω_{ij}. Etant donnée la transformation infinitésimale

$$(\delta y) = (\omega)\,(x),$$

cherchons sa transformée par une transformation de g. On aura:

$$(\delta y') = (b)\,(\delta y) = (b)\,(\omega)\,(x) = (b)\,(\omega)\,(a)^{-1}\,(x').$$

La transformation infinitésimale cherchée est donc définie par la matrice (ω'):

$$(\omega') = (b)\,(\omega)\,(a)^{-1} = (b)\,(\omega)\,(\bar{a})^*.$$

La matrice (ω) est donc transformée comme la matrice dont les éléments sont les produits $y_i\,\bar{x}_j$. Posons

$$(a) = e^{i\varphi}\,(a_1), \qquad (b) = e^{i\psi}\,(b_1),$$

les matrices (a_1) et (b_1) étant unimodulaires. On aura: $e^{i(k+1)\varphi + i(q+1)\psi} = 1$.

$$(3) \qquad (w') = e^{i(\psi-\varphi)}\,(b_1)\,(\omega)\,(\bar{a}_1)^* = e^{i\theta}\,(b_1)\,(\omega)\,(\bar{a}_1)^*.$$

Le groupe linéaire d'isotropie est le groupe linéaire γ qui opère sur les paramètres ω_{ij} et $\bar{\omega}_{ij}$ et qui est définie par l'équation (3). Il est du type considéré à la fin du §5. Le groupe linéaire qui opère sur les ω_{ij} seuls est le produit de groupe d'équation $(\omega') = e^{i\theta}\,(\omega)$ et du groupe γ'' défini par:

$$(\omega') = (b_1)\,(\omega)\,(\bar{a}_1)^*.$$

Désignons par $\mathfrak{g}_k$ et $\mathfrak{g}_q$ les groupes linéaires unimodulaires unitaires opérant respectivement sur les variables $x_0, \cdots, x_k$ et sur les variables $y_0, \cdots, y_q$. *Le groupe γ'' est identique au groupe suivant lequel sont transformés les produits $y_i\,x_j$ lorsque les variables x_j et y_i subissent respectivement les transformations des groupes $\mathfrak{g}_k$ et $\mathfrak{g}_q$.*

7. Détermination des invariants intégraux et des nombres de Betti. Déterminons les invariants intégraux de l'espace riemannien symétrique réalisé par la variété de Grassmann V. D'après le §5, il suffit de déterminer les formes extérieures à coefficients constants des variables ω_{ij} et $\bar{\omega}_{ij}$, ces formes étant invariantes par γ. Il n'y a pas de forme extérieure invariante de degré impair, c'est-à-dire *les nombres de Betti relatifs aux dimensions impaires sont nuls.* Pour chercher les formes invariantes de degré $2s$, nous décomposons en groupes irréductibles le groupe γ''_s qui opère sur les formes extérieures $[\omega_{i_1 j_1} \cdots \omega_{i_s j_s}]$

lorsque les variables ω_{ij} subissent les transformations de γ''. Nous employons la *méthode des poids dominants de M. E. Cartan.*[11]

γ'' est une représentation linéaire irréductible du groupe semi-simple $\mathfrak{g}_k \times \mathfrak{g}_q$, produit direct de $\mathfrak{g}_k$ et de $\mathfrak{g}_q$. Nous considérons le sous-groupe abélien de γ'' correspondant aux transformations infinitésimales:

$$\delta x_j = \lambda_j x_j, \qquad (j = 0, 1, \cdots, k)$$

$$\delta y_i = \mu_i y_i, \qquad (i = 0, 1, \cdots, q)$$

où les λ_j et les μ_i sont imaginaires pures et satisfont aux relations $\sum_j \lambda_j = 0$ et $\sum_i \mu_i = 0$. Par rapport à ce sous-groupe abélien, la variable x_j est de poids λ_j, y_i est de poids μ_i et ω_{ij} est de poids $\lambda_j + \mu_i$. La variable $[\omega_{i_1 j_1} \cdots \omega_{i_s j_s}]$ est de poids $\lambda_{j_1} + \cdots + \lambda_{j_s} + \mu_{i_1} + \cdots + \mu_{i_s}$. Ce poids est une forme linéaire à coefficients entiers:

$$\Pi(\lambda, \mu) = m_0 \lambda_0 + \cdots + m_k \lambda_k + n_0 \mu_0 + \cdots + n_q \mu_q$$

où $m_0 + m_1 + \cdots + m_k = n_0 + n_1 + \cdots + n_q = s$. Le poids $\Pi(\lambda, \mu)$ est dit *plus haut* qu'un autre poids $\Pi'(\lambda, \mu)$ défini par des entiers m_j' et n_i' lorsque:

$$m_j - m_k \geqq m_j' - m_k', \qquad (j = 0, 1, \cdots, k-1)$$

$$n_i - n_q \geqq n_i' - n_q', \qquad (i = 0, 1, \cdots, q-1),$$

l'une au moins de ces conditions étant une inégalité. D'après un théorème fondamental de M. E. Cartan, *tout groupe irréductible Γ_s contenu dans γ_s'' est complètement déterminé par sa variable dominante.* Pour qu'une variable u, c'est-à-dire une combinaison linéaire de formes extérieures $[\omega_{i_1 j_1} \cdots \omega_{i_s j_s}]$, soit la variable dominante d'un groupe irréductible Γ_s, il faut et il suffit que le poids de u soit plus haut que le poids de n'importe quelle variable transformée de u.

Au lieu de considérer seulement les transformations des groupes unitaires $\mathfrak{g}_k$ et $\mathfrak{g}_q$, appliquons aux variables x_j et y_i les transformations des deux groupes linéaires unimodulaires à coefficients complexes arbitraires. Les produits $x_j y_i$ subissent alors les transformations d'un groupe semi-simple γ'''. C'est le groupe à paramètres complexes qui a la même structure que le groupe unitaire γ''. Toute représentation linéaire Γ de γ'' peut être élargie en une représentation linéaire Γ' de γ'''. Si Γ' est irréductible, Γ l'est également, et réciproquement.[12] Appliquons donc aux variables ω_{ij} les transformations du groupe γ'''. Pour que u soit une variable dominante, il faut et il suffit que son poids soit plus haut que le poids de toute variable transformée. Il suffit de considérer les transformations infinitésimales de γ''' qui correspondent aux transformations infinitésimales $x_{j'} \partial f/\partial x_j$ et $y_{i'} \partial f/\partial y_i$. A une variable de poids Π elles font correspondre une variable de poids $\Pi + \lambda_{j'} - \lambda_j$ ou $\Pi + \mu_{i'} - \mu_i$. Ainsi pour que

[11] Voir E. Cartan, *f* et H. Weyl, *a*.

[12] Voir H. Weyl, *a*, p. 289–290.

u soit variable dominante, il faut et il suffit que les transformées de u par $x_{j'}\ \partial f/\partial x_j$ ou $y_{i'}\ \partial f/\partial y_i$ soient identiquement nulles lorsque $j' < j$ ou $i' < i$.

Soit u une combinaison linéaire de formes extérieures $[\omega_{i_0 j_1} \cdots \omega_{i_s j_s}]$. Pour que u soit une variable de poids $\Pi(\lambda, \mu)$, il faut et il suffit qu'elle ne soit composée que de formes extérieures de poids $\Pi(\lambda, \mu)$. Si $\Pi(\lambda, \mu)$ est le poids dominant d'un groupe irréductible Γ_s, on a:

$$m_0 \geqq m_1 \geqq \cdots \geqq m_k, \qquad n_0 \geqq n_1 \geqq \cdots \geqq n_q,$$

car en permutant les λ_j entre eux et les μ_i entre eux, on obtient encore des poids de Γ_s. En permutant convenablement les ω_{ij} dans chacune des formes extérieures $[\omega_{i_0 j_1} \cdots \omega_{i_s j_s}]$, on peut d'abord écrire les m_0 variables ω_{ij} dont l'indice j est égal à 0, puis les m_1 variables ω_{ij} dont l'indice j est égal à 1, etc. On arrangera les variables ω_{ij} qui correspondent à une même valeur de j de telle façon que leurs indices i soient rangés par ordre croissant de gauche à droite. u aura alors la forme suivante:

$$u = \Sigma \alpha_P P_{(i)} [\omega_{i_1 0} \cdots \omega_{i_{m_0} 0} \omega_{i_{m_0+1} 1} \omega_{i_{m_0+2} 1} \cdots].$$

Dans cette expression, $P_{(i)}$ désigne une permutation P à effectuer sur les indices $i_1, i_2, \cdots, i_s$, et α_P est un coefficient dépendant de la permutation P. Nous allons ranger les termes de cette expression dans un ordre bien déterminé. Prenons d'abord les termes dans lesquels le premier indice i a la plus petite valeur. Parmi ceux-ci prenons les termes dans lesquels le deuxième indice i a la plus petite valeur. En continuant la sélection d'après le même principe, on arrive finalement à un terme bien déterminé, qui sera pris comme premier terme. Le deuxième terme sera obtenu de la même façon parmi les termes restants. On voit qu'on pourra ainsi ranger tous les termes.

Soit $[\omega_{i_1 0} \cdots \omega_{i_{m_0} 0} \omega_{i_{m_0+1} 1} \cdots]$ le premier terme. Nous allons montrer qu'il est égal à la forme extérieure

$$[\omega_{0\,0}\, \omega_{1\,0} \cdots \omega_{m_0-1\,0}\, \omega_{0\,1}\, \omega_{1\,1} \cdots \omega_{m_1-1\,1} \cdots],$$

que nous designons par v. Soit i_h le premier des indices $i_1, \cdots, i_{m_0}$ tel que $i_h > h - 1$. Posons $i_h = p$ et appliquons la transformation infinitésimale $y_{h-1}\ \partial f/\partial y_p$. La variable transformée de u doit être identiquement nulle. Or c'est la somme des termes qu'on déduit des termes de u en remplaçant par $h - 1$ un quelconque des indices i qui est égal à p. En particulier, en remplaçant dans le premier terme de u l'indice i_h par $h - 1$, on obtient une forme extérieure qui ne peut figurer qu'une fois dans la variable transformée de u. Par conséquent il n'y a pas d'indice i_h tel que $i_h > h - 1$. Ce raisonnement montre que le premier terme de u est égal à v. Or v est une variable dominante de poids $\Pi(\lambda, \mu)$. Donc $u - v$ est une variable dominante de poids $\Pi(\lambda, \mu)$. Ainsi le deuxième terme de u devrait être un multiple de v, c'est-à-dire la variable u se réduit simplement à la forme extérieure v.

A tout ensemble d'entiers $(m_0, m_1, \cdots, m_k)$, tel que

$$m_0 + m_1 + \cdots + m_k = s, \qquad q + 1 = n - k \geqq m_0 \geqq m_1 \geqq \cdots \geqq m_k \geqq 0$$

correspond une variable dominante, $[\omega_{00}\omega_{10}\cdots\omega_{m_0-1\,0}\omega_{01}\cdots]$, et par suite un groupe irréductible Γ_s bien déterminé. A deux ensembles distincts $(m_0, m_1, \cdots, m_k)$ et $(m_0', m_1', \cdots, m_k')$ correspondent deux groupes irréductibles Γ_s et Γ_s' qui ne sont pas équivalents. Donc la valeur moyenne du carré du module du caractère de γ_s'' est égal au nombre de ces ensembles d'entiers $(m_0, m_1, \cdots, m_k)$. Donc:

THÉORÈME. *Etant donné l'espace riemannien symétrique réalisé par la variété de Grassmann V, le nombre de ses invariants intégraux de degré $2s$ est égal au nombre d'ensembles de $k+1$ entiers $(m_0, m_1, \cdots, m_k)$ tels que $m_0 + m_1 + \cdots + m_k = s$ et $n - k \geqq m_0 \geqq m_1 \geqq \cdots \geqq m_k \geqq 0$. Ce nombre est aussi le nombre de Betti R_{2s} de la variété V.*

Pour connaître les variables qui sont transformées par le groupe Γ_s correspondant aux entiers $(m_0, m_1, \cdots, m_k)$, on peut utiliser le schéma $(m_0, m_1, \cdots, m_k)$ qui caractérise d'après M. H. Weyl,[13] la représentation linéaire irréductible du groupe $\mathfrak{g}_k$ de poids dominant $m_0\lambda_0 + m_1\lambda_1 + \cdots + m_k\lambda_k$.

1	2	.	.	.	m_0
m_0+1	.	.	m_0+m_1		
.	.	.			

schéma $(m_0, m_1, \cdots, m_k)$

Le schéma définit un certain arrangement des s premiers nombres entiers Dans la première ligne nous écrivons les m_0 premiers nombres, dans la deuxième ligne les m_1 nombres suivants, etc. Le coefficient n_i du poids dominant de Γ_s est le nombre d'entiers contenus dans la $i^{\text{ième}}$ colonne du schéma. En échangeant les lignes avec les colonnes, on obtient un deuxième schéma qui caractérise la représentation linéaire de $\mathfrak{g}_q$ de poids dominant $n_0\mu_0 + n_1\mu_1 + \cdots + n_q\mu_q$. Γ_s est équivalent au produit des deux représentations linéaires correspondant à ces deux schémas.

Soit P une permutation des nombres du schéma $(m_0, m_1, \cdots, m_k)$. Soit A une permutation par laquelle les nombres de chaque colonne sont permutés entre eux. Soit B une permutation par laquelle les nombres de chaque ligne sont permutés entre eux. Si P peut se mettre sous la forme AB (on effectue d'abord la permutation B, ensuite la permutation A), nous posons ϵ_P = signe de la permutation A; sinon $\epsilon_P = 0$. Si P peut se mettre sous la forms BA, nous posons ϵ_P' = signe de B, sinon $\epsilon_P' = 0$. Les variables du groupe Γ_s sont alors les quantités $\sum_P \epsilon_P P_{(j)}[\omega_{i_1 j_1}\cdots\omega_{i_s j_s}]$. On a en effet:

$$\sum_P \epsilon_P P_{(j)}[\omega_{i_1 j_1}\cdots\omega_{i_s j_s}] = \sum_P \epsilon_P' P_{(i)}[\omega_{i_1 j_1}\cdots\omega_{i_s j_s}].$$

On en conclut que les quantités $\sum_P \epsilon_P P_{(j)}[\omega_{i_1 j_1}\cdots\omega_{i_s j_s}]$ sont transformées entre

[13] Voir H. Weyl, *a*.

elles par les operations de Γ_s qui correspondent soit aux transformations de $\mathfrak{g}_k$, soit aux transformations de $\mathfrak{g}_q$. Par suite ces quantités sont bien transformées entre elles par le groupe Γ_s.

Le groupe Γ_s étant clos, il laisse invariante une forme d'Hermite définie positive. En remplaçant dans cette forme d'Hermite les produits ordinaires par des produits extérieurs, on obtient une forme extérieure invariante par γ. La forme d'Hermite est déterminée à un facteur constant près, car il n'y a qu'une forme extérieure invariante par Γ_s. L'invariant intégral le plus général est défini par une combinaison linéaire arbitraire des formes extérieures invariantes qui correspondent aux différents groupes Γ_s.

Ainsi l'invariant intégral de second ordre est défini par $\Sigma[\omega_{ij}\bar{\omega}_{ij}]$. Il est facile d'en donner l'expression explicite en fonction des coordonnées plückeriennes $p_{\alpha_0\alpha_1\dots\alpha_k}$. Le groupe hermitien elliptique G transforme les variables $p_{\alpha_0\alpha_1\dots\alpha_k}$ suivant un groupe linéaire irréductible clos, qui laisse invariante une certaine forme d'Hermite. Comme chaque terme de cette forme doit avoir un poids nul par rapport aux transformations infinitésimales $\delta x_i = \lambda_i x_i$, où $i = 0, 1, \cdots, n$ et où les λ_i sont imaginaires pures, on trouve que la forme d'Hermite invariante est $\Sigma p_{\alpha_0\alpha_1\dots\alpha_k}\bar{p}_{\alpha_0\alpha_1\dots\alpha_k}$. Imposons aux variables $p_{\alpha_0\alpha_1\dots\alpha_k}$ la condition $\Sigma p_{\alpha_0\alpha_1\dots\alpha_k}\bar{p}_{\alpha_0\alpha_1\dots\alpha_k} = 1$. L'invariant intégral du second degré sera $\int\Sigma[dp_{\alpha_0\alpha_1\dots\alpha_k}\,d\bar{p}_{\alpha_0\alpha_1\dots\alpha_k}]$. On en déduit un invariant intégral de degré $2s$, à savoir $\int(\Sigma[dp_{\alpha_0\alpha_1\dots\alpha_k}\,d\bar{p}_{\alpha_0\alpha_1\dots\alpha_k}])^s$. En général il y aura encore d'autres invariants intégraux de degré $2s$.

Si V est l'espace projectif $[n]$ lui-même, c'est-à-dire si $k = 0$, on obtient ainsi tous les invariants intégraux. On a $R_{2s} = 1$. On peut imposer aux coordonnées $x_0, \cdots, x_n$ la condition $\Sigma x_i\bar{x}_i = 1$. L'invariant intégral de degré $2s$ sera $\int(\Sigma[dx_i d\bar{x}_i])^s$.

VÉRIFICATION. Donnons une vérification des résultats précédents en utilisant un théorème[14] de topologie dû à M. S. Lefschetz. D'après ce théorème, le nombre de points fixes d'une déformation d'une multiplicité orientable V_n est égal à la caractéristique d'Euler-Poincaré multipliée par $(-1)^n$. La variété de Grassmann V est une multiplicité orientable, parce que c'est une variété algébrique sans singularités, on encore parce que le groupe d'isotropie g est connexe. D'après les résultats précédents, la caractéristique d'Euler-Poincaré est égale à la somme des nombres de Betti R_{2s}. C'est le nombre d'ensembles d'entiers $(m_0, m_1, \cdots, m_k)$ tels que $n - k \geqq m_0 \geqq m_1 \geqq \cdots \geqq m_k = 0$. En remplaçant m_i par $l_i = m_i + k - i$, on obtient les ensembles d'entiers $(l_0, l_1, \cdots, l_k)$ tels que $n \geqq l_0 > l_1 > \cdots > l_k \geqq 0$. La caractéristique d'Euler-Poincaré est donc égale à $\binom{n+1}{k+1}$.

D'autre part, le groupe projectif complexe étant connexe, toute transformation projective de $[n]$ est une déformation de $[n]$. Etant donnée une transformation

[14] Voir S. Lefschetz, *a*, p. 272.

projective générale, le nombre de points fixes est $n+1$ et le nombre d'éléments $[k]$ fixes est $\binom{n+1}{k+1}$. On retrouve bien la valeur de la caractéristique d'Euler-Poincaré. Cependant il reste à démontrer que tout élément fixe doit être compté avec l'ordre de multiplicité égal à $+1$. Ceci résulte du théorème général suivant:

THÉORÈME. *Etant donné un groupe continu de Lie G opérant transitivement dans une variété analytique complexe V, les transformations de G étant définies par des équations analytiques par rapport à des variables et des paramètres complexes, tout point fixe isolé d'une transformation de G doit être compté avec un ordre de multiplicité égal à $+1$.*

En effet, soit O un point fixe isolé d'une transformation T de G. Un système de paramètres canoniques de G sera fourni par un système de r paramètres complexes $e_1, \cdots, e_r$. Supposons que les $r-n$ dernières transformations infinitésimales de base engendrent le groupe d'isotropie relatif à O. L'ensemble des n paramètres complexes $e_1, e_2, \cdots, e_n$ constitue un système de coordonnées pour un voisinage v_0 de O. (Voir §1.) Dans v_0 la transformation T est représentée par des équations linéaires par rapport aux variables $e_1, e_2, \cdots, e_n$. Pour définir la multiplicité du point fixe O, on considère le produit topologique $v_0 \times v_0$, c'est-à-dire le domaine défini par le système de coordonnées $e_1, e_2, \cdots, e_n, e_1', e_2', \cdots, e_n'$. Dans ce système de coordonnées, la transformation T et la transformation identique sont représentées par des variétés planes complexes qui se coupent seulement au point $e_i = e_i' = 0$. L'indice de Kronecker de ce point d'intersection est égal à $+1$, ce qui prouve le théorème énoncé.

III. Etude directe de la topologie des variétés de Grassmann[15]

8. **Démonstration d'un lemme.** Comme les variétés de Grassmann ont une définition géométrique très simple, il est naturel d'étudier leurs propriétés topologiques sans passer par l'intermédiaire des intégrales de différentielles exactes. On peut chercher à les décomposer en *cellules*. Il n'est pas nécessaire d'arriver à une subdivision en *simplexes*; en ce qui concerne l'homologie et l'homotopie, des cellules d'une espèce plus générale peuvent remplir le même rôle que les simplexes. L'intérieur des cellules que nous aurons à considérer sera homéomorphe à l'intérieur d'une sphère ordinaire, mais la frontière ne sera pas nécessairement homéomorphe à la surface de cette sphère. L'emploi de cellules de cette espèce est justifié à cause du lemme suivant:

LEMME. *Etant donnés un complexe K et un sous-complexe L dans K, si $K-L$ est homéomorphe à une cellule ouverte, toute chaîne sur K, de dimension inférieure à la dimension de $K-L$, peut être déformée d'une façon continue en une chaîne sur L. Pendant la déformation les points de L peuvent être maintenus fixes.*

L'expression "*cellule ouverte*" désignera toujours un ensemble de points

[15] On trouvera un résumé de cette méthode et l'indication des variétés auxquelles nous l'avons appliquée dans une note des Comptes Rendus: C. Ehresmann, *b*.

homéomorphe à l'intérieur d'une sphère ordinaire. Les complexes considérés sont formés de simplexes. Si L est un sous-complexe du complexe K, nous appelons K–voisinage de L l'ensemble des simplexes de K dont la frontière a une partie commune avec L. Soit N_K^L ce K–voisinage. $K-N_K^L$ est l'ensemble des éléments de K qui n'ont pas de partie commune avec L. $K-N_K^L$ est un complexe fermé, N_K^L est un ensemble ouvert. Le K–voisinage est dit *normal*, si tout simplexe de K qui a tous ses sommets sur L est situé entièrement sur L. Si le K–voisinage n'est pas normal, on peut considérer le complexe dérivé K' de K et le K'–voisinage de L sera normal.[16] Nous pouvons donc supposer dans la suite que le K–voisinage de L est normal.

Si K–L est homéomorphe à une cellule, toute chaîne sur K–L est déformable en une chaîne sur N_K^L, à condition que la dimension de la chaîne soit inférieure à celle de K–L. Soit, en effet, C_p une chaîne sur K, c'est-à-dire C_p est l'image continue et uniforme d'une chaîne formée de simplexes à p dimensions. D'après le théorème fondamental[17] de la déformation des chaînes sur un complexe, la chaîne C_p peut être déformée en une sous-chaîne de K. Cette opération effectuée, nous supposerons que C_p ne contient pas tous les simplexes de K–L, ce qui a lieu en particulier si p est inférieur à la dimension de K–L. Il existe alors un point O de K–L qui est extérieur à C_p. Considérons l'homéomorphie représentant K–L sur l'intérieur d'une sphère S. Il existe une sphère S' intérieure à S et contenant à son intérieur l'image $\bar{O}$ du point O ainsi que l'image de l'ensemble fermé $K-N_K^L$. On effectue alors la déformation suivante de l'image $\bar{C}_p$ de la partie de la chaîne C_p intérieure à K–L: Les points de $\bar{C}_p$ extérieurs à S' et situés sur S' sont laissés fixes. Les points de $\bar{C}_p$ intérieurs à S' sont déplacés le long des rayons issus de O et sont finalement projetés sur S'. Après cette déformation la chaîne C_p se trouvera complètement à l'intérieur de N_K^L.

Dans le raisonnement précédent, on aurait pu considérer aussi bien le K'–voisinage $N_{K'}^L$, où K' est une subdivision normale de K. Nous supposerons donc C_p déformé en une chaîne sur $N_{K'}^L$. Il existe une déformation continue du complexe K qui réduit $N_{K'}^L$ à L et qui réduit par suite C_p à une chaîne sur L. En effet, comme le K–voisinage de L est normal, tout simplexe E_q de N_K^L a en commun avec L un simplexe E'_q, bien déterminé. Pour faire la subdivision normale de K, nous introduisons un sommet M à l'intérieur de E_q et un sommet M' à l'intérieur de $E'_{q'}$. Les points M sont les sommets des simplexes de $N_{K'}^L$. Le segment rectiligne $M\,M'$ est bien défini à l'intérieur du simplexe E_q. Faisons correspondre à tout point M le point M' correspondant tandis que les sommets de $K-N_K^L$ se correspondent à eux-mêmes. Nous définissons ainsi une transformation barycentrique du complexe K' et les segments $M\,M'$ définissent une déformation barycentrique qui réduit $N_{K'}^L$ à L. Seuls les simplexes de N_K^L-L subissent une déformation; en particulier les points de L restent fixes.

[16] Comparer S. Lefschetz, *a*, p. 91.

[17] Voir S. Lefschetz, *a*, p. 86.

Il existe donc bien une déformation continue satisfaisant à l'énoncé du lemme. En particulier la frontière de la cellule ouverte $K-L$ est toujours connexe.

Le fait qu'une chaîne contenue dans un voisinage N^L de L peut être déformée en une chaîne sur L, a été démontré par M. S. Lefschetz[18] avec des hypothèses moins restrictives. Il suffit de supposer qu'il existe une métrique dans K et que K et L sont clos et localement connexes. Le lemme démontré ici est valable si l'on suppose de plus que $K-L$ est une cellule ouverte. Mais nous n'aurons qu'à considérer le cas où K et L sont des complexes.

9. **Conséquences du lemme.** Soit K un complexe formé de simplexes. Les points d'un simplexe qui n'appartiennent pas à la chaîne-frontière de celui-ci seront appelés points intérieurs du simplexe. L'ensemble de plusieurs simplexes sera considéré comme une cellule lorsque l'ensemble de leurs points intérieurs est homéomorphe à l'intérieur d'une sphère. Les simplexes qui composent une telle cellule forment une multiplicité combinatoire ouverte, en vertu d'un théorème de M. E. R. van Kampen. La frontière de la cellule sera aussi composée de simplexes. On peut fixer une orientation de la cellule, ce qui détermine en même temps une orientation de ses simplexes de dimension maximum. La cellule orientée définit alors une chaîne bien déterminée, somme de ses simplexes orientés, et par suite une chaîne-frontière bien déterminée.

Supposons que le complexe K puisse être considéré comme un assemblage de cellules du type considéré satisfaisant aux conditions suivantes: 1. *Tout point de K est intérieur à une cellule et une seule*; 2. *La frontière de chaque cellule est un ensemble de points formé par la somme d'un certain nombre de cellules de dimension inférieure.* Désignons par E_p^i les cellules orientées à p dimensions ou également les chaînes de simplexes définies par ces cellules orientées. Comme la chaîne-frontière d'une cellule E_p^i est un cycle orienté, elle contient toute la chaîne définie par une cellule E_{p-1}^j, dès qu'elle en contient un simplexe. D'une façon plus précise, les chaînes-frontières des cellules E_p^i sont des combinaisons linéaires des chaînes E_{p-1}^j, ce qu'on écrit sous la forme:

$$E_p^i \rightarrow \Sigma\, \mu_{ij}^{p-1}\, E_{p-1}^j. \tag{1}$$

Soit K_p le complexe formé par toutes les cellules de K dont la dimension est inférieure ou égale à p. Le lemme du paragraphe précédent entraine alors la conséquence suivante:

a. Toute chaîne C_p sur le complexe K peut être déformée d'une façon continue en une chaîne sur le complexe K_p.

Le preuve par induction est évidente.

En particulier tout cycle Γ_p sur K peut être déformé en un cycle sur K_p. Une nouvelle déformation de ce cycle fournit ensuite un cycle formé de simplexes de K_p. D'après une remarque déjà faite, le cycle sera déformé en une

[18] Voir S. Lefschetz, *a*, p. 92–94.

combinaison linéaire des chaînes E_p^i plus des simplexes dégénérés le cas échéant. Donc:

b. Tout cycle Γ_p est homologue à une combinaison linéaire des cellules E_p^i.

Soit Γ_p un cycle homologue à zéro. Il existe une chaîne C_{p+1} telle que: $C_{p+1} \to \Gamma_p$. Supposons Γ_p déformé en une combinaison linéaire de cellules E_p^i. D'après le lemme, nous pouvons laisser fixes les points de Γ_p et déformer C_{p+1} en une chaîne C'_{p+1} sur K_{p+1}. Nous pouvons supposer que C'_{p+1} est formé de simplexes de K_{p+1}. Comme C'_{p+1} est un cycle (mod K_p), en d'autres termes comme sa frontière est sur K_p, C'_{p+1} sera une combinaison linéaire de cellules E_{p+1}^i. L'homologie $\Gamma_p \sim 0$ sera une combinaison linéaire des homologies $\Sigma \mu_{ij}^p E_p^j \sim 0$. Par conséquent:

c. Les bases d'homologie, les nombres de Betti et les coefficients de torsion s'obtiennent simplement par la réduction des systèmes de relations (1) *à la forme canonique habituelle.*[19]

Les considérations précédentes permettent d'étudier la structure topologique d'un grand nombre de variétés algébriques. Rappelons que toute variété algébrique, réelle ou complexe, admet une subdivision en simplexes.[20] Un domaine ouvert d'une variété analytique sera appelé une cellule analytique, s'il est homéomorphe à l'intérieur d'une sphère et si sa frontière est composée d'éléments analytiques. L'ensemble d'une cellule analytique et de sa frontière peut être subdivisé en simplexes de telle façon que le complexe formé par cette subdivision admet un sous-complexe recouvrant exactement la frontière. Donc le lemme du paragraphe précédent s'applique et toute chaîne C_q sur la cellule analytique à p dimensions peut être déformée en une chaîne sur la frontière de la cellule, à condition que q soit inférieur à p. Comme nous n'aurons qu'à considérer des variétés algébriques, il sera commode d'utiliser l'expression "*cellule algébrique*" pour désigner un *domaine ouvert d'une variété algébrique, ce domaine étant homéomorphe à une cellule ouverte et sa frontière étant composée d'éléments de variétés algébriques.* Un point sera considéré comme une cellule algébrique de dimension zéro.

Etant donnée une variété algébrique V, supposons qu'on sache trouver un système de variétés algébriques contenues dans V et fournissant une subdivision de V en cellules algébriques E_p^i. Cette subdivision devra satisfaire aux conditions suivantes: 1. Tout point de V appartient à une cellule E et une seule; 2. La frontière de chaque cellule E_p^i est la somme d'un certain nombre de cellules de dimension inférieure à p. Il existe dans ces conditions un complexe formé de simplexes recouvrant la variété V et tel que chaque cellule algébrique E_p^i soit subdivisée par un sous-complexe de ce complexe. Alors le lemme du paragraphe précédent est applicable et le complexe formé des cellules algébriques E_p^i jouit des propriétés *a*, *b* et *c*.

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[19] Ceci est à rapprocher d'un résultat de topologie combinatoire qui indique dans quelles conditions un agrégat de cellules peut être traité comme une cellule unique dans le calcul des nombres de Betti et des coefficients de torsion. Voir A. W. Tucker, *a*.

[20] Voir B. L. van der Waerden, *a*.

Les variétés algébriques considérées ici peuvent être supposées réelles ou complexes. En ce qui concerne les variétés algébriques *complexes*, on arrive ainsi à des propriétés topologiques particulièrement simples. Une cellule algébrique complexe a un nombre pair de dimensions réelles; sa frontière, qui est par hypothèse formée de parties de variétés algébriques complexes, sera au plus à $2p - 2$ dimensions réelles, si la cellule est à $2p$ dimensions réelles. Une cellule algébrique complexe définit donc un cycle. On sait qu'une variété algébrique complexe V est toujours orientable et qu'il existe une orientation naturelle bien déterminée pour V et pour toute variété algébrique contenue dans V.[21] Supposons que par un système de variétés algébriques complexes on sache décomposer la variété V en cellules algébriques complexes E^i_{2p}. La structure topologique de V sera alors très simple et très particulière. Chaque cellule E^i_{2p} définira un cycle orienté bien déterminé. Tout cycle Γ_{2p+1} sur V pourra être déformé en un cycle Γ'_{2p+1} sur le complexe K_{2p} défini par l'ensemble des cellules algébriques dont la dimension est inférieure ou égale à $2p$. Ce cycle Γ'_{2p+1} est uniquement formé d'éléments dégénérés; *donc tout cycle* Γ_{2p+1} *est homologue à zéro.* Il n'y aura aucune homologie liant les cycles algébriques définis par les cellules E^i_{2p}. *Donc ces cycles formeront une base minima du groupe d'homologie relatif à la dimension* $2p$ *et il n'y aura pas de coefficients de torsion.*

Une cellule algébrique à une dimension complexe a pour frontière un point unique. Donc si la variété V est connexe, le complexe de ses cellules algébriques n'a qu'un élément E_0, formé par un point unique. Toute courbe fermée dans V peut être réduite à ce point par déformation continue. *Le groupe de Poincaré se réduit à l'identité.*

Un exemple classique d'une variété algébrique de cette nature est fourni par *l'espace projectif complexe* $[n]$. Il suffit de considérer une suite de variétés planes $[n-1], [n-2], \cdots, [2], [1], [0]$ satisfaisant aux conditions:

$$[n] \supset [n-1] \supset \cdots \supset [p] \supset [p-1] \supset \cdots \supset [2] \supset [1] \supset [0].$$

Les cellules algébriques sont $[p] - [p-1]$. Si $[p]$ a pour équations

$$x_{p+1} = x_{p+2} = \cdots = x_n = 0$$

une chaîne C_s, où $s \leqq 2p+1$, peut être déformée en une chaîne sur $[p]$ par la déformation homographique:

$$x'_i = x_i \qquad (i = 0, 1, \cdots, p)$$

$$x'_j = \lambda x_j, \qquad (j = p+1, p+2, \cdots, n)$$

le facteur λ variant d'une façon continue de 1 à O; par une petite déformation préalable il faudrait pourtant faire disparaître les points d'intersection éventuels de C_s avec la variété plane d'équations: $x_0 = x_1 = \cdots = x_p = 0$.

[21] Voir S. Lefschetz, *a*, ch. VII.

Nous allons voir que la réduction en cellules algébriques réussit pour un grand nombre de variétés algébriques; en particulier l'étude de la topologie des variétés de Grassmann devient par là d'une simplicité extrême.

10. **Les bases d'homologie des variétés de Grassmann.** Nous considérons la variété V de tous les $[k]$ d'un espace projectif complexe $[n]$. Pour éviter les confusions, nous représentons par $\{k\}$ une variété plane $[k]$ qui est considérée comme élément générateur d'une certaine variété. Introduisons les symboles de Schubert $[a_0, a_1, \cdots, a_k]$. Etant données $k+1$ variétés $[a_0], [a_1], \cdots, [a_k]$ satisfaisant aux conditions:

$$0 \leqq a_0 < a_1 < \cdots < a_k \leqq n, \qquad [a_0] \subset [a_1] \subset \cdots \subset [a_k] \subset [n],$$

le symbole $[a_0, a_1, \cdots, a_k]$ représente la variété des $\{k\}$ contenus dans $[a_k]$ et ayant en commun avec $[a_i]$ une variété plane à i dimensions au moins. Les variétés $[a_0, a_1, \cdots, a_k]$ seront appelées *variétés fondamentales de Schubert.*[22]

Ainsi $[0, 1, \cdots, k]$ représente un élément $\{k\}$ unique; la variété V elle-même a pour symbole $[n-k, n-k-1, \cdots, n]$.

La variété fondamentale $[a_0, a_1, \cdots, a_k]$ est une variété algébrique contenue dans V. On montre facilement que sa dimension complexe est $a_0 + a_1 + \cdots + a_k - \dfrac{k(k+1)}{2}$. La variété $[a_i]$ qui intervient dans la définition de $[a_0, a_1, \cdots, a_k]$ est parfaitement déterminée, sauf dans le cas où $a_{i+1} - a_i = 1$. Un $\{k\}$ sera un élément singulier de la variété $[a_0, a_1, \cdots, a_k]$, s'il coupe l'un des $[a_i]$ suivant un $[i+1]$, pourvu qu'on ait $a_{i+1} - a_i > 1$. Le lieu des éléments singuliers sera composé d'un certain nombre de variétés fondamentales dont les symboles s'obtiennent à partir de $[a_0, a_1, \cdots, a_k]$ en exprimant que l'élément $\{k\}$ coupe suivant un $[i+1]$ l'une des variétés $[a_i]$ pour laquelle $a_{i+1} - a_i > 1$. Par exemple, le lieu des éléments singuliers de $[1, 3, 4, 6]$ est composé de $[0, 1, 4, 6]$ et $[1, 2, 3, 4]$. Une variété fondamentale moins le lieu de ses éléments singuliers est une variété homogène transformée transitivement par un groupe projectif de l'espace $[n]$. Les seules variétés fondamentales qui n'ont pas d'éléments singuliers sont les variétés de tous les $\{k\}$ contenus dans un $[m]$, ou les variétés de tous les $\{k\}$ contenus dans $[m]$ et passant par $[r]$. Ce sont des variétés de Grassmann; la variété des $\{k\}$ contenus dans $[m]$ et passant par $[r]$ est, en effet, en correspondance biunivoque avec la variété des $[k-r-1]$ d'un espace $[m-r-1]$.

Ceci étant, nous allons montrer que la variété V peut être décomposée en cellules algébriques par un système de variétés fondamentales de Schubert. Prenons dans $[n]$ une suite de variétés planes $[n-1], [n-2], \cdots, [1], [0]$ satisfaisant aux conditions:

$$(1) \qquad [n] \supset [n-1] \supset [n-2] \supset \cdots \supset [1] \supset [0]$$

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[22] *H. Schubert, a* et *b*; *C. Segre, a*, p. 795.

et considérons toutes les variétés $[a_0, a_1, \cdots, a_k]$ définies à l'aide des $[a_i]$ de cette suite. Démontrons par récurrence la règle suivante:

La variété $[a_0, a_1, \cdots, a_k]$ *devient homéomorphe à une cellule ouverte, lorsqu'on en retranche toutes les variétés fondamentales dont les symboles se déduisent du symbole* $[a_0, a_1, \cdots, a_k]$ *en y remplaçant un des nombres* a_i *par* $a_i - 1$. *Si un symbole obtenu de cette façon n'a pas de sens, on le néglige.*

Supposons la règle démontrée pour les variétés fondamentales engendrées par des $\{k-1\}$ et montrons qu'elle est alors également vérifiée pour les variétés fondamentales engendrées par des $\{k\}$. Comme elle est vérifiée pour $k = 0$ (espace projectif), elle sera démontrée pour tous les cas.

Considérons la variété $[a_0, a_1, \cdots, a_k]$. Un élément général $\{k\}$ de cette variété rencontre $[a_0]$ en un point M et coupe suivant un $\{k-1\}$ un hyperplan général $[n-1]'$. Nous pouvons choisir l'hyperplan $[n-1]'$ de telle façon qu'il satisfait aux conditions suivantes: $[a_0]$ est coupé par $[n-1]'$ suivant $[a_0 - 1]$; les variétés $[a_0 + 1]$, $[a_0 + 2]$, $\cdots$, $[n-1]$ de la suite (1) sont coupées par $[n-1]'$ respectivement suivant des variétés $[a_0]'$, $[a_0 + 1]'$, $\cdots$, $[n-2]'$ distinctes des variétés planes de la suite (1). L'élément $\{k\}$ de $[a_0, a_1, \cdots, a_k]$ correspond d'une façon biunivoque à l'ensemble d'un point M de $[a_0]$ et d'un $\{k-1\}$ de l'hyperplan $[n-1]'$, pourvu que $\{k\}$ ne rencontre pas la variété $[a_0 - 1]$. L'élément $\{k-1\}$ n'est pas arbitraire dans $[n-1]'$. En effet, si $\{k\}$ coupe $[a_i]$ suivant un $[i]$, il coupe $[a_i - 1]'$ suivant un $[i-1]$. Réciproquement, si $\{k\}$ coupe $[a_i - 1]'$ suivant un $[i-1]$, il coupera $[a_i]$ suivant un $[i]$, à condition que le point d'intersection M de $\{k\}$ et de $[a_0]$ ne soit pas sur $[a_0 - 1]$. Donc les $\{k\}$ qui ne rencontrent pas $[a_0 - 1]$ correspondent aux $\{k-1\}$ de la variété $[a_1 - 1, a_2 - 1, \cdots, a_k - 1]' - [a_0 - 1, a_2 - 1, \cdots, a_k - 1]'$, définie à l'aide des variétés planes de la suite $[a_0 - 1]$, $[a_0]'$, $[a_0 + 1]'$, $\cdots$, $[n-1]'$. Or par hypothèse la variété $[a_1 - 1, a_2 - 1, \cdots, a_k - 1]'$ se réduit à une cellule ouverte lorsqu'on en retranche toutes les variétés dont le symbole se déduit du symbole $[a_1 - 1, a_2 - 1, \cdots, a_k - 1]'$ en remplaçant un des nombres $a_i - 1$ par $a_i - 2$. Lorsque $\{k-1\}$ décrit cette cellule ouverte pendant que M décrit la cellule ouverte $[a_0] - [a_0 - 1]$, l'élément $\{k\}$ déterminé par l'ensemble de $\{k-1\}$ et de M engendre une variété homéomorphe au produit topologique des deux cellules ouvertes; c'est-à-dire $\{k\}$ engendre une cellule ouverte.

La cellule ouverte ainsi engendrée par $\{k\}$ est la variété $[a_0, a_1, \cdots, a_k]$ dont on a enlevé les variétés suivantes: 1°. La variété $[a_0 - 1, a_1, \cdots, a_k]$ pour laquelle le point M se trouve sur $[a_0 - 1]$. 2°. Les variétés des $\{k\}$ qui coupent l'hyperplan $[n-1]'$ suivant les $\{k-1\}$ générateurs des variétés $[a_1 - 1, a_2 - 1, \cdots, a_i - 2, \cdots, a_k - 1]'$, où i est l'un quelconque des indices $1, 2, \cdots, k$, le point M étant arbitraire sur $[a_0]$. Si $\{k-1\}$ coupe $[a_0 + s]'$ suivant un $[r]$, l'élément $\{k\}$, déterminé par $\{k-1\}$ et un point M extérieur à $[a_0 - 1]$, coupe $[a_0 + s + 1]$ suivant un $[r+1]$. Donc la variété $[a_1 - 1, a_2 - 1, \cdots, a_i - 2, \cdots, a_k - 1]'$ correspond à la variété $[a_0, a_1, a_2, \cdots, a_i - 1, \cdots, a_k]$ engendrée par des $\{k\}$. Certains symboles qu'on serait ainsi conduit à écrire peuvent ne pas avoir de sens. Par hypothèse, nous

n'avons pas à tenir compte des symboles $[a_1 - 1, a_2 - 1, \cdots, a_i - 2, \cdots, a_k - 1]'$ qui n'ont pas de sens. Si $a_0 = a_1 - 1$, le symbole $[a_0, a_1 - 1, \cdots, a_k]$ n'a pas de sens. On voit qu'il suffit de le remplacer également par zéro.

Le raisonnement précédent suppose que $a_0 > 0$. Il reste à considérer le cas d'une variété $[a_0, a_1, \cdots, a_k]$, où $a_r = r$ et $a_{r+1} > r + 1$. Soit $[n - r - 1]'$ une variété qui ne coupe pas $[a_r]$ et qui coupe $[a_r + 1]$, $[a_r + 2]$, $\cdots$, $[n - 1]$ suivant des variétés $[0]'$, $[1]'$, $\cdots$, $[n - r - 2]'$. Tout $\{k\}$ de la variété $[a_0, a_1, \cdots, a_k]$ est bien déterminé par son intersection avec $[n - r - 1]'$. Cette intersection est un élément $\{k - r - 1\}$ engendrant la variété

$$[a_{r+1} - r - 1, \cdots, a_i - r - 1, \cdots, a_k - r - 1]'$$

définie à l'aide des variétés de la suite $[0]'$, $[1]'$, $\cdots$, $[n - r - 1]'$. Comme la règle énoncée est vérifiée pour cette variété engendrée par des $\{k - r - 1\}$, on voit qu'elle est encore vérifiée pour la variété $[a_0, a_1, \cdots, a_k]$. Ceci prouve la règle dans tous les cas.

D'après cette règle, *l'ensemble des variétés fondamentales* $[a_0, a_1, \cdots, a_k]$ *fournit une subdivision de la variété de Grassmann V en cellules algébriques.* Nous avons un complexe de cellules algébriques auquel nous pouvons appliquer les résultats du §9.

La variété V est une multiplicité orientable à $(k + 1)(n - k)$ dimensions complexes. Chaque variété fondamentale $[a_0, a_1, \cdots, a_k]$ définit un cycle avec une orientation bien déterminée; ce cycle sera représenté par le même symbole $[a_0, a_1, \cdots, a_k]$. Nous pouvons énoncer les théorèmes:

THÉORÈME. *Tout cycle* Γ_{2s} *ou* Γ_{2s+1} *sur la variété V peut être déformé en un cycle qui recouvre complètement un certain nombre de variétés fondamentales* $[a_0, a_1, \cdots, a_k]$, *où* $a_0 + a_1 + \cdots + a_k - \dfrac{k(k+1)}{2} \leqq s$. *En particulier, la variété V est simplement connexe.*

THÉORÈME. *Les cycles algébriques* $[a_0, a_1, \cdots, a_k]$, *à s dimensions complexes, forment une base minima du groupe d'homologie pour la dimension* $2s$. *Il n'y a pas de coefficients de torsion. Les nombres de Betti relatifs aux dimensions impaires sont tous nuls.*

11. **Intersection de deux cycles de dimensions complémentaires.** Nous dirons que deux cycles sur la variété de Grassmann V ont des dimensions complémentaires, lorsque la somme de leurs dimensions est égale à la dimension réelle de V, c'est-à-dire égale à $2(k + 1)(n - k)$. Ainsi deux variétés $[a_0, a_1, \cdots, a_k]$ et $[b_0, b_1, \cdots, b_k]$ sont de dimensions complémentaires lorsque:

$$(1) \qquad a_0 + a_1 + \cdots + a_k + b_0 + b_1 + \cdots + b_k = (k + 1)n.$$

Etant données deux variétés $[a_0, a_1, \cdots, a_k]$ et $[a_0, a_1, \cdots, a_k]'$ dont les symboles sont composés des mêmes nombres, il existe une transformation projective de $[n]$, et par suite une déformation continue, qui transforme la première variété en la deuxième. Ces deux variétés définissent toujours des cycles algébriques

homologues. Soient $[a_0, a_1, \cdots, a_k]$ et $[b_0, b_1, \cdots, b_k]$ deux variétés fondamentales de dimensions complémentaires. Par une transformation projective, amenons la variété $[a_0, a_1, \cdots, a_k]$ en une position telle que les variétés $[a_0], \cdots, [a_k]$ aient des positions générales par rapport aux variétés $[b_0], \cdots, [b_k]$ qui servent à définir $[b_0\ b_1, \cdots, b_k]$. Pour qu'il existe alors un élément d'intersection de $[a_0, a_1, \cdots, a_k]$ et de $[b_0, b_1, \cdots, b_k]$, il faut qu'il existe un $\{k\}$ qui coupe $[a_i]$ suivant un $[i]$ et qui coupe $[b_{k-i}]$ suivant un $[k-i]$. Or les variétés $[i]$ et $[k-i]$ ont un point d'intersection sur $\{k\}$; donc $[a_i]$ et $[b_{k-i}]$ ont un point d'intersection. Comme $[a_i]$ a une position générale par rapport à $[b_{k-i}]$, il en résulte que $a_i + b_{k-i} \geqq n$. Mais en vertu de l'équation (1) on a alors $a_i + b_{k-i} = n$, quel que soit i, et le symbole de la variété $[b_0, b_1, \cdots, b_k]$ est $[n-a_k, \cdots, n-a_1, n-a_0]$. Réciproquement les deux variétés $[a_0, a_1, \cdots, a_k]$ et $[n-a_k, \cdots, n-a_1, n-a_0]$ ont un élément d'intersection et un seul, si l'on a donné à la première variété une position générale par rapport à la deuxième. Cet élément d'intersection est l'élément $\{k\}$ déterminé par les $k+1$ points d'intersection $[a_i] \cdot [n-a_i]'$, en supposant que $[a_i]$ et $[n-a_i]'$ servent à définir respectivement les deux variétés fondamentales considérées.

Le cycle d'intersection de deux cycles quelconques sur une multiplicité a été défini par M. S. Lefschetz.[23] Dans le cas de deux cycles de dimensions complémentaires, on peut définir le *nombre algébrique de points d'intersection* ou *l'indice de Kronecker des deux cycles*. Si A et B sont deux variétés algébriques de dimensions complémentaires contenues dans la multiplicité K formée par une variété algébrique sans singularités, l'ordre de multiplicité d'un point d'intersection isolé de A et de B est le nombre algébrique de points qu'il représente quand on évalue l'indice de Kronecker des cycles algébriques A et B. Rappelons que cet ordre de multiplicité est positif lorsque K, A et B sont des variétés complexes.

Soit M un point d'intersection isolé de A avec B, les variétés A, B et K étant supposées complexes. M admet dans K un voisinage formé par une cellule analytique complexe E, dont les points sont définis par un système de coordonnées complexes $u_1, u_2, \cdots, u_\alpha$. A l'intérieur de E, les voisinages de M sur les deux variétés A et B sont des domaines de variétés complexes qui sont représentées par des variétés analytiques complexes a et b dans le domaine des coordonnées $u_1, u_2, \cdots, u_\alpha$. Si le point M est un point ordinaire de a et de b, c'est-à-dire s'il admet sur chacune des deux variétés sécantes un voisinage formant une cellule analytique complexe, l'ordre de multiplicité de M est égal à $+1$, à condition que M soit le seul point d'intersection des deux plans tangents en M aux variétés a et b. Dans le cas général on fera subir à b une petite déformation définie par une translation arbitraire dans le domaine des variables $u_1, u_2, \cdots, u_\alpha$. Après cette déformation, les variétés a et b se couperont en h points d'intersection voisins de M et tels qu'en chacun de ces points

[23] Voir S. Lefschetz, *a*, ch. IV.

l'intersection sera du type simple qu'on vient de considérer. L'ordre de multiplicité de M sera alors égal à h.

Ceci posé, il est facile de démontrer d'une façon rigoureuse que *l'indice de Kronecker des deux variétés fondamentales* $[a_0, a_1, \cdots, a_k]$ *et* $[n - a_k, \cdots, n - a_1, n - a_0]$ *est égal à* $+1$. Désignons comme au §6 par $x_0, x_1, \cdots, x_k, y_0, y_1, \cdots, y_q$ les coordonnées homogènes des points de $[n]$, et soit $[k]_0$ la variété définie par les équations: $y_0 = y_1 = \cdots = y_q = 0$. Un élément $\{k\}$ voisin de $\{k\}_0$ a pour équations:

$$y_i = \omega_{ij} x_j \qquad \begin{pmatrix} i = 0, 1, \cdots, q \\ j = 0, 1, \cdots, k \end{pmatrix}. \tag{2}$$

C'est l'élément défini par $k + 1$ points $P_0, P_1, \cdots, P_k$, le point P_j ayant pour coordonnées

$$x_j = 1, \qquad x_{j'} = 0, \qquad y_i = \omega_{ij} \qquad (j' \neq j).$$

Les équations (2) sont valables tant que $\{k\}$ ne rencontre pas la variété $[n - k - 1]$ d'équations: $x_0 = x_k = \cdots = x_k = 0$. Les $(k + 1)(n - k)$ quantités ω_{ij} forment un système de coordonnées complexes pour la cellule ouverte $[n - k, n - k + 1, \cdots, n] - [n - k - 1, n - k + 1, \cdots, n]$.

Considérons une variété $[a_0, a_1, \cdots, a_k]$ contenant l'élément $\{k\}_0$. Nous supposons que $\{k\}_0$ est un élément général de cette variété c'est-à-dire que son intersection avec la variété $[a_r]$ qui intervient dans la définition de $[a_0, a_1, \cdots, a_k]$ est une variété plane à r dimensions exactement. Par une transformation projective de $[n]$ laissant invariant l'élément $\{k\}_0$, on pourra amener les variétés $[a_r]$ en coïncidence avec les variétés suivantes:

$$[a_k] \qquad y_{m_0} = y_{(m_0+1)} = \cdots = y_q = 0 \qquad a_k = m_0 + k$$

$$[a_{k-1}] \qquad x_0 = y_{m_1} = y_{(m_1+1)} = \cdots = y_q = 0 \qquad a_{k-1} = m_1 + k - 1$$

$$[a_{k-r}] \qquad x_0 = x_1 = \cdots = x_{r-1} = y_{m_r} = y_{(m_r+1)} = \cdots = y_q = 0 \qquad a_{k-r} = m_r + k - r$$

$$[a_0] \qquad x_0 = x_1 = \cdots = x_{k-1} = y_{m_k} = y_{(m_k+1)} = \cdots = y_q = 0 \qquad a_0 = m_k$$

Les inégalités $n \geqq a_k > a_{k-1} > \cdots > a_1 > a_0 \geqq 0$ entrainent les inégalités $n - k \geqq m_0 \geqq m_1 \geqq \cdots \geqq m_k \geqq 0$. Soit $\{k\}$ un élément général de $[a_0, a_1, \cdots, a_k]$ défini par les équations (2). La condition que les variétés planes $\{k\}$ et $[a_{k-r}]$ se coupent suivant une variété plane à $k - r$ dimensions entraine $\omega_{ij} = 0$ pour tous les ensembles d'indices i et j vérifiant les inégalités $i \geqq m_r$, $j \geqq r$. Dans l'espace des variables complexes ω_{ij}, la variété $[a_0, a_1, \cdots, a_k]$ est donc représentée par une variété plane dont les équations sont:

$$\omega_{ij} = 0 \qquad \text{pour } i \geqq m_j.$$

Ceci prouve, en passant, que la dimension complexe de la variété $[a_0, a_1, \cdots, a_k]$ est

$$m_0 + m_1 + \cdots + m_k = a_0 + a_1 + \cdots + a_k - \frac{k(k+1)}{2}.$$

Considérons maintenant la variété d'éléments $\{k\}$ ayant pour équations: $\omega_{ij} = 0$ pour $i < m_j$. C'est une variété fondamentale de symbole $[n - a_k, \cdots, n - a_1, n - a_0]$ définie par les variétés planes suivantes:

$$[n - a_0] \qquad y_0 = y_1 = \cdots = y_{(m_k - 1)} = 0$$

$$[n - a_1] \qquad x_k = y_0 = y_1 = \cdots = y_{(m_{k-1} - 1)} = 0$$

$$[n - a_{k-r}] \qquad x_{r+1} = \cdots = x_k = y_0 = y_1 = \cdots = y_{m_r} = 0$$

$$[n - a_k] \qquad x_1 = \cdots = x_k = y_0 = y_1 = \cdots = y_{m_0} = 0.$$

Les deux variétés $[a_0, a_1, \cdots, a_k]$ et $[n - a_k, \cdots, n - a_1, n - a_0]$ sont représentées dans l'espace des variables ω_{ij} par deux variétés planes qui n'ont qu'un point d'intersection, $\omega_{ij} = 0$; c'est le point représentatif de l'élément $\{k\}_0$. Cet élément doit donc être compté avec l'ordre de multiplicité égal à $+1$ dans l'intersection des deux variétés. Comme il n'y a pas d'autre élément d'intersection, l'indice de Kronecker des deux variétés est égal à $+1$:

$$[a_0, a_1, \cdots, a_k] \cdot [n - a_k, \cdots, n - a_1, n - a_0] = +1,$$

en désignant par $A \cdot B$ l'indice de Kronecker de deux cycles A et B.

Pour chaque dimension $2s$ nous avons une base minima du groupe d'homologie formée par R_{2s} cycles fondamentaux $[a_0, a_1, \cdots, a_k]$, où $a_0 + a_1 + \cdots + a_k - \dfrac{k(k+1)}{2} = s$. La matrice des nombres d'intersection des cycles fondamentaux de dimension $2s$ avec les cycles fondamentaux de dimension complémentaire est égale à la matrice-unité, à condition d'écrire ces cycles dans un ordre convenable. Soit Γ_{2s} un cycle quelconque sur V. On a l'homologie:

$$\Gamma_{2s} \sim \Sigma(n - a_k, \cdots, n - a_1, n - a_0)\,[a_0, a_1, \cdots, a_k]. \tag{3}$$

Le coefficient $(n - a_k, \cdots, n - a_1, n - a_0)$ dans cette homologie est le nombre d'intersections de Γ_{2s} et de $[n - a_k, \cdots, n - a_1, n - a_0]$:

$$(n - a_k, \cdots, n - a_1, n - a_0) = \Gamma_{2s} \cdot [n - a_k, \cdots, n - a_1, n - a_0].$$

Soit $\Gamma'_{2s'}$ un cycle de dimension complémentaire, $s' = (k + 1)(n - k) - s$. On aura:

$$\Gamma'_{2s'} \sim \Sigma(a_0, a_1, \cdots, a_k)'\,[n - a_k, \cdots, n - a_1, n - a_0],$$

où:

$$(a_0, a_1, \cdots, a_k)' = \Gamma'_{2s'} \cdot [a_0, a_1, \cdots, a_k].$$

Le nombre d'intersections de Γ_{2s} et de $\Gamma'_{2s'}$ sera:

$$\Gamma_{2s} \cdot \Gamma'_{2s'} = \Sigma(n - a_k, \cdots, n - a_1, n - a_0)(a_0, a_1, \cdots, a_k)'. \tag{4}$$

En particulier si Γ_{2s} est le cycle défini par une variété algébrique A, les coefficients $(n - a_k, \cdots, n - a_1, n - a_0)$ sont des nombres positifs. H. Schubert, qui les a déjà considérés dans ses recherches de géométrie énumérative, les a appelés les *degrés* (en allemand *Gradzahlen*) de la variété A. Si l'on a deux variétés algébriques A et B de dimensions complémentaires, les degrés de A

étant les nombres $(n - a_k, \cdots, n - a_1, n - a_0)$ et ceux de B étant les nombres $(a_0, a_1, \cdots, a_k)'$, le nombre d'intersections de A et de B sera:

$$(4)' \qquad A \cdot B = \Sigma(n - a_k, \cdots, n - a_1, n - a_0)(a_0, a_1, \cdots, a_k)'.$$

Cette formule constitue une généralisation du *théorème de Bézout.* En effet, soit V *l'espace projectif complexe* $[n]$, qui est un cas particulier des variétés de Grassmann. Si $A_{(s)}$ est une variété algébrique contenue dans $[n]$, de dimension complexe s, on a l'homologie: $A_{(s)} \sim (n - s)[s]$. Le nombre $(n - s)$ est le degré de A d'après la définition ordinaire de cette notion. Soit $B_{(n-s)}$ une variété algébrique à $n - s$ dimensions complexes. On a: $B_{(n-s)} \sim (s)'[n - s]$, ou $(s)'$ est le degré de $B_{(n-s)}$. Il en résulte: $A_{(s)} \cdot B_{(n-s)} = (n - s)(s)'$, ce qui constitue le théorème de Bézout.

Considérons aussi le cas particulier des *variétés de droites.* V est la variété des droites de $[n]$, $A_{(s)}$ et $B_{(s')}$ sont des variétés algébriques engendrées par des droites. Supposons $s + s' = 2n - 2$. Alors les formules précédentes deviennent:

$$A_{(s)} \sim \Sigma(n - q, n - p)\,[p, q], \qquad p + q = s$$

$$B_{(s')} \sim \Sigma(p, q)'\,[n - q, n - p]$$

$$A_{(s)} \cdot B_{(s')} = \Sigma(n - q, n - p)\,(p, q)'.$$

Un complexe de droites, c'est-à-dire une variété algébrique $A_{(2n-3)}$ contenue dans V, n'a qu'un degré: $A_{(2n-3)} \sim (0, 2)\,[n - 2, n]$, où $(0, 2) = A_{(2n-3)} \cdot [0, 2]$. Si $(0, 2) = 1$, $A_{(2n-3)}$ est un complexe linéaire. Une surface réglée, c'est-à-dire une variété $B_{(1)}$, a aussi un seul degré; c'est le nombre $(n - 2, n) = B_1 \cdot [n - 2, n]$. Le nombre de droites communes à $A_{(2n-3)}$ et $B_{(1)}$ est égal au produit de leurs degrés.

Les bases d'homologie de la variété des droites de l'espace projectif [3] sont:

[2, 3]		$R_3 = 1$
[1, 3]		$R_6 = 1$
[0, 3]	[1, 2]	$R_4 = 2$
[0, 2]		$R_2 = 1$
[0, 1]		$R_0 = 1.$

Considérons deux congruences de droites $A_{(2)}$ et $B_{(2)}$. On a:

$$A_{(2)} \sim (0, 3)\,[0, 3] + (1, 2)\,[1, 2]$$

$$B_{(2)} \sim (0, 3)'\,[0, 3] + (1, 2)'\,[1, 2].$$

Le nombre de droites communes sera:

$$A_{(2)} \cdot B_{(2)} = (0, 3)(0, 3)' + (1, 2)(1, 2)'.$$

Cette égalité constitue le théorème de Halphen relatif aux congruences de droites.

12. **Remarque au sujet du problème des caractéristiques de Schubert.** La formule (4)′ du paragraphe précédent exprime un résultat de géométrie énumérative déjà obtenu par Schubert.[24] Cependant le raisonnement de Schubert ne peut pas être considéré comme rigoureux, car il ne s'appuie pas sur une définition précise et générale de l'ordre de multiplicité d'un point d'intersection et il manque de précision en ce qui concerne une certaine déformation de variétés algébriques.

Rappelons que la topologie fournit une interprétation et une justification du *calcul symbolique de Schubert.*[25] Elle permet aussi de montrer que le *problème des caractéristiques* pour une variété algébrique sans singularités admet toujours une solution.

Etant donnée la variété de Grassmann V engendrée par les $[k]$ de l'espace $[n]$, les bases d'homologie pour les cycles algébriques sont fournies par les variétés $[a_0, a_1, \cdots, a_k]$. Soit $A_{(s)}$ une variété algébrique à s dimensions contenue dans V. Elle définit un cycle algébrique $A_{(s)}$. La relation (3) du paragraphe précédent entraîne une égalité symbolique de Schubert:

$$A_{(s)} = \Sigma(n - a_k, \cdots, n - a_1, n - a_0)[a_0, a_1, \cdots, a_k].$$

Cette égalité fournit la solution du problème des caractéristiques. Les différentes variétés $[a_0, a_1, \cdots, a_k]$ ne sont liées par aucune égalité symbolique de Schubert. *Dans une variété de Grassmann l'égalité symbolique de Schubert* $A_{(s)} = B_{(s)}$ *et l'homologie* $A_{(s)} \sim B_{(s)}$ *sont donc deux relations équivalentes.*

13. **Remarque à propos d'un théorème de M. F. Severi.** Nos résultats topologiques sont à rapprocher d'un théorème démontré par M. F. Severi. Considérons la variété de Grassmann engendrée par les $\{k\}$ de $[n]$ et soit $V_{(d)}$ sa représentation à l'aide des coordonnées plückeriennes $p_{i_0 i_1 \cdots i_k}$ dans l'espace projectif $[r]$, où $r = \binom{n+1}{k+1} - 1$. Pour la dimension complexe $d - 1$, la base d'homologie est formée par la variété $[n - k - 1, n - k + 1, \cdots, n]$ dont l'équation peut se ramener à $p_{01 \cdots k} = 0$. C'est une section hyperplane particulière de $V_{(d)}$. Soit C une section de $V_{(d)}$ par un $[r - 1]$ général. La variété $[0, 1, \cdots, k - 1, k + 1]$ est représentée sur $V_{(d)}$ par une droite. Donc $C \cdot [0, 1, \cdots, k - 1, k + 1] = 1$ et par suite $C \sim [n - k - 1, n - k + 1, \cdots, n]$. Un complexe algébrique, $A_{(d-1)}$, est une variété algébrique à $d - 1$ dimensions complexes contenue dans $V_{(d)}$. On aura $A_{(d-1)} \sim \lambda C$, où λ est le degré du complexe. D'après un théorème de M. S. Lefschetz,[26] cette homologie entraîne l'équivalence algébrique $A_{(d-1)} \equiv \lambda C$. Ainsi C forme la base minima de Severi,

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[24] Voir H. Schubert, *a*. Une autre démonstration se trouve dans F. Severi, *c*. Il y a de nombreux travaux de géométrie énumérative concernant les variétés fondamentales de Schubert. On pourra consulter à ce sujet C. Segre, *a*.

[25] Voir B. L. van der Waerden, *a*. Voir également F. Severi, *b*.

[26] S. Lefschetz, *b*, p. 25 et 48.

et les complexes de degré λ forment un système continu complet. Comme le nombre de Betti relatif à la dimension 1 est nul, la variété $V_{(d)}$ est régulière et par suite $A_{(d-1)}$ et λC sont des variétés génériques d'un même système linéaire. Donc *les complexes algébriques de degré* λ *forment un système linéaire complet* qui contient évidemment le système linéaire des intersections complètes de $V_{(d)}$ par les hypersurfaces de degré λ de $[r]$. Dans son mémoire consacré aux variétés de Grassmann, M. F. Severi[27] démontre une propriété plus précise, à savoir que tout complexe algébrique est l'intersection complète de $V_{(d)}$ par une hypersurface de $[r]$.

IV. Etude des autres familles de variétés algébriques de la classe considérée

14. **La quadrique complexe non dégénérée à n dimensions complexes.** Nous nous proposons de montrer comment notre méthode élémentaire permet d'étudier la topologie des variétés algébriques qui sont classées au §5 avec les variétés de Grassmann comme fournissant des espaces homogènes symétriques.[28]

Rappelons quelques propriétés classiques d'une quadrique non dégénérée Q_n appartenant à un espace $[n+1]$.[29] En supposant $n = 2p$ ou $n = 2p+1$, la quadrique Q_n contient une infinité de variétés planes à p dimensions, mais ne contient pas de variétés planes de dimension supérieure à p. Si $n = 2p+1$, ces génératrices planes à p dimensions forment une seule famille continue. Si $n = 2p$, les génératrices planes à p dimensions forment deux familles distinctes. Deux génératrices $[p]$ et $[p]'$ de même famille se coupent suivant une variété plane à $p - 2s$ dimensions. Deux génératrices $[p]$ et $[p]'$ qui n'appartiennent pas à la même famille se coupent suivant une variété plane à $p - 2s - 1$ dimensions. s est un entier tel que $p - 2s \geqq -1$ et $p - 2s - 1 \geqq -1$. On convient de considérer une intersection nulle comme une intersection de dimension -1. En général, deux génératrices $[p]$ et $[p]'$ de même système (de systèmes différents) ont un point commun si p est pair (impair), et n'ont pas de point commun si p est impair (pair).

Soit O un point de la quadrique non dégénérée Q_n. Soit $[n]$ l'hyperplan tangent en O. $[n]$ coupe Q_n suivant un cône Q_{n-1} de sommet O. Soit $[n]'$ un hyperplan ne passant pas par O et coupant $[n]$ suivant $[n-1]'$. Par projection centrale de centre O on met $Q_n - Q_{n-1}$ en correspondance biunivoque avec la cellule ouverte $[n]' - [n-1]'$. *Donc* $Q_n - Q_{n-1}$ *est une cellule algébrique ouverte.*

Soit $[k]$ une génératrice plane de Q_n. Supposons $k < p$. $[k]$ admet une variété conjuguée $[n-k]$ qui coupe Q_n suivant une quadrique Q_{n-k-1}; celle-ci est $k+1$ fois dégénérée et admet $[k]$ comme lieu de ses points doubles. Soit O' un point de Q_{n-k-1} non situé sur $[k]$. L'ensemble de O' et de $[k]$ détermine une génératrice $[k+1]$ dont la variété conjuguée est une variété $[n-k-1]$.

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[27] Voir F. Severi, *a*.

[28] Les résultats relatifs aux quadriques complexes sont connus. Voir E. Cartan, *g* et B. L. van der Waerden, *b*.

[29] Voir C. Segre, *a*.

Q_n est coupé par $[n-k-1]$ suivant une quadrique Q_{n-k-2}. Tout point de $Q_{n-k-1} - Q_{n-k-2}$ correspond d'une façon biunivoque à une droite de $[n-k]$ qui passe par O' et qui n'est pas située dans $[n-k-1]$. *Donc* $Q_{n-k-1} - Q_{n-k-2}$ *est une cellule algébrique ouverte.*

Supposons $n = 2p+1$. Prenons une génératrice $[p]$ de Q_{2p+1} et considérons dans $[p]$ des variétés $[p-1], \cdots, [1], [0]$ telles que $[p] \supset [p-1] \supset \cdots \supset [1] \supset [0]$. Nous faisons correspondre à une variété $[k]$ de cette suite la quadrique Q_{n-k-1}, intersection de Q_n par la variété conjuguée de $[k]$. Si $k = p$, Q_{n-p-1} se réduit à $[p]$. On a ainsi une suite de variétés:

$$Q_{2p+1} \supset Q_{2p} \supset \cdots \supset Q_{p+1} \supset [p] \supset [p-1] \supset \cdots \supset [1] \supset [0].$$

La différence entre deux variétés successives forme une cellule algébrique ouverte. *Ces variétés forment les bases d'homologie. Il n'y a pas de coefficients de torsion. L'indice de Kronecker* $[k]\cdot Q_{n-k}$ *est égal à* 1.

Supposons $n = 2p$. Considérons sur Q_{2p} la suite des variétés planes:

$$[p] \supset [p-1] \supset \cdots \supset [1] \supset [0].$$

A une variété $[k]$ de cette suite nous faisons encore correspondre une quadrique Q_{n-k-1}. Pour $k = p-1$, Q_{n-k-1} se réduit à deux génératrices $[p]$ et $[p]'$, de systèmes différents, qui se coupent suivant $[p-1]$. *Les bases d'homologie sont encore formées par* $Q_{2p}, Q_{2p-1}, \cdots, Q_{p+1}, [p], [p]', [p-1], \cdots, [1], [0]$. Remarquons qu'il y a deux variétés de base pour la dimension p. *Il n'y a pas de coefficients de torsion.* On a encore: $[k]\cdot Q_{n-k} = 1$ pour $k \neq p$. Si p est impair: $[p]\cdot[p]' = 1$, $[p]\cdot[p] = [p]'\cdot[p]' = 0$. Si p est pair:

$$[p]\cdot[p] = [p]'\cdot[p]' = 1, \qquad [p]\cdot[p]' = 0.$$

15. **La variété des génératrices planes à p dimensions d'une quadrique complexe non dégénérée Q_{2p}.** Considérons sur la quadrique complexe non dégénérée Q_{2p} les génératrices planes à p dimensions appartenant à l'un des deux systèmes de génératrices à p dimensions. Elles engendrent une variété algébrique V à $p(p+1)/2$ dimensions complexes. Un élément quelconque de V sera designé par $\{p\}$. Nous nous proposons de subdiviser V en cellules algébriques.

Supposons d'abord $p = 2$. Les points de Q_4 représentent les droites complexes d'un espace projectif [3]. Une génératrice plane de Q_4 représente les droites de [3] qui passent par un point ou les droites de [3] qui sont situées dans un plan. Les éléments $\{2\}$ de la variété V sont donc en correspondance biunivoque avec les points de [3]. Cette remarque conduit aux résultats suivants: Soit $\{2\}_0$ un élément particulier de V. C'est une variété plane $[2]_0$ contenue dans Q_4. Prenons dans la variété $[2]_0$ une droite [1] et un point [0] sur cette droite. Tout élément $\{2\}$ de V a un point en commun avec $[2]_0$. Considérons la variété des éléments de V qui rencontrent la droite [1]. Nous la représentons par $[1, \cdot, \cdot]$. La variété des éléments de V qui contiennent le

point [0] sera représentée par [0, ·, ·]. Représentons par [0, 1, 2] l'élément particulier $\{2\}_0$ et par [2, ·, ·] la variété totale V. On reconnait alors que [2, ·, ·] − [1, ·, ·] est une cellule algébrique ouverte. Il en est de même pour [1, ·, ·] − [0, ·, ·] et pour [0, ·, ·] − [0, 1, 2].

Passons à l'étude du cas général. Considérons sur la quadrique Q_{2p} une variété plane $[p]_0$ formant un élément particulier $\{p\}_0$ de la variété V lorsque p est pair, et n'appartenant pas à V lorsque p est impair. L'intersection d'un élément $\{p\}$ de V avec $[p]_0$ a toujours un nombre pair de dimensions; en général ce sera un point. Introduisons dans $[p]_0$ une suite de variétés $[p-1], [p-2], \cdots, [0]$ telles que:

$$[p]_0 \supset [p-1] \supset [p-2] \supset \cdots \supset [0]. \tag{1}$$

Considérons $2i+1$ variétés de cette suite: $[a_0] \subset [a_1] \subset \cdots \subset [a_{2i}]$. Le symbole $[a_0, a_1, a_2, \cdots, a_{2i}, \cdot, \cdot\cdot]$ représentera la variété des éléments $\{p\}$ de V qui coupent $[a_s]$ suivant une variété à s dimensions. Dans le symbole nous mettons $p-2i$ points à la suite du dernier nombre a_{2i} pour indiquer que l'élément générateur est une variété plane à p dimensions. La variété V elle-même est représentée par $[p, \cdot, \cdots\cdot]$. Nous allons démontrer le lemme suivant:

LEMME. *La variété* $[a_0, a_1, \cdots, a_{2i}, \cdot, \cdot\cdot]$ *devient une cellule algébrique ouverte lorsqu'on en retranche les variétés représentées par les symboles qu'on obtient en diminuant d'une unité un quelconque des nombres du symbole donné, ou en remplaçant, dans le cas où* $a_{2i} < p-1$, *les deux premiers points qui suivent* a_{2i} *par* $p-1$ *et* p. *Tout symbole qui n'a pas de sens devra être remplacé par zéro.*

Le lemme est bien vérifié pour $p=1$ et $p=2$. En le supposant exact pour une quadrique Q_{2p-2}, nous allons le démontrer pour une quadrique Q_{2p}.

La quadrique Q_{2p} appartient à un espace $[2p+1]$. Soit $[2p]'$ un hyperplan ne passant pas par le point [0] de la suite (1). L'hyperplan tangent à Q_{2p} au point [0] coupe $[2p]'$ suivant une variété $[2p-1]'$. L'intersection de Q_{2p} avec $[2p-1]'$ est une quadrique non dégénérée Q_{2p-2}. Soit $\{p\}$ une génératrice de Q_{2p} appartenant à la variété V. Si $\{p\}$ ne passe pas par [0], la projection stéréographique de centre [0] lui fait correspondre d'une façon biunivoque une variété plane $\{p\}'$ de l'hyperplan $[2p]'$. $\{p\}'$ coupe $[2p-1]'$ suivant une génératrice $\{p-1\}'$ de la quadrique Q_{2p-2}. $\{p-1\}'$ engendre un des systèmes de génératrices de Q_{2p-2}. Tout $\{p\}'$ appartenant à $[2p]'$ et coupant $[2p-1]'$ suivant une telle génératrice $\{p-1\}'$ correspond à une génératrice $\{p\}$ ne passant pas par [0].

Les variétés de la suite (1) sont coupées par $[2p-1]'$ suivant des variétés $[p-1]'_0, [p-2]', \cdots, [0]'$ telles que:

$$[p-1]'_0 \supset [p-2]' \supset \cdots \supset [0]'. \tag{2}$$

Considérons la variété $[a_0, a_1, \cdots, a_{2i}, \cdot, \cdot\cdot]$ et supposons $a_0 > 0$. Les $\{p\}$ de cette variété qui ne contiennent pas [0] se projettent suivant les $\{p\}'$ de $[2p]'$ qui coupent $[2p-1]'$ suivant des génératrices $\{p-1\}'$ de Q_{2p-2}. Ces génératrices $\{p-1\}'$ engendrent la variété $[a'_0, a'_1, \cdots, a'_{2i}, \cdot, \cdot\cdot]'$ définie à

l'aide des variétés de la suite (2), le nombre a_s' étant égal à $a_s - 1$. Nous pouvons appliquer le lemme à la variété $[a_0', a_1', \cdots, a_{2i}', \cdot, \cdot\cdot]'$. Celle-ci devient une cellule algébrique ouverte A lorsqu'on en retranche les variétés indiquées dans l'énoncé du lemme. Il reste à démontrer que les $\{p\}'$ de l'hyperplan $[2p]'$ qui coupent $[2p-1]'$ suivant les éléments $\{p-1\}'$ de A forment une cellule ouverte.

Les nombres $a_0', a_1', \cdots, a_{2i}'$ sont des nombres de la suite $0, 1, \cdots, p-1$; désignons par $b_0', b_1', \cdots, b_j'$ les autres nombres de cette suite. Considérons sur Q_{2p-2} une génératrice $[p-1]_0''$ qui coupe $[p-1]_0'$ suivant une variété $[j]$. Nous choisissons $[p-1]_0''$ de telle façon que $[j]$ appartienne à la variété fondamentale de Schubert de symbole $[b_0', b_1', \cdots, b_j']$ définie à l'aide de variétés de la suite (2). Nous supposons que $[j]$ occupe une position générale dans $[b_0', b_1', \cdots, b_j']$. Les variétés $[p-1]_0'$ et $[p-1]_0''$ sont des génératrices de Q_{2p-2} qui n'appartiennent pas au même système, car leur intersection est à $j = p - 1 - (2i+1)$ dimensions. Soit $\{p-1\}'$ un élément quelconque de la cellule ouverte A. Montrons que $\{p-1\}'$ n'a pas de point commun avec $[p-1]_0''$. On voit d'abord que $\{p-1\}'$ n'a pas de point commun avec $[j]$; sinon $\{p-1\}'$ appartiendrait à une des variétés-frontières de la cellule A. Soit $[2p-2-j]$ la variété conjuguée de $[j]$ par rapport à la quadrique Q_{2p-2}. Les variétés $[2p-2-j]$ et $\{p-1\}'$ ne peuvent pas appartenir à un même hyperplan de l'espace $[2p-1]'$, comme leurs variétés conjuguées n'ont pas de point commun. Donc l'intersection de $[2p-2-j]$ et $\{p-1\}'$ est à $p-2-j = 2i$ dimensions; c'est précisément l'intersection de $\{p-1\}'$ avec $[p-1]_0'$. La variété $[p-1]_0''$ appartient à $[2p-2-j]$. Il en résulte bien que $\{p-1\}$ et $[p-1]_0''$ n'ont pas de point commun.

Soit $[p]_0''$ une variété contenue dans $[2p]'$ et coupant $[2p-1]'$ suivant $[p-1]_0''$. Soit $\{p\}'$ une variété plane à p dimensions contenue dans $[2p]'$ et coupant $[2p-1]'$ suivant $\{p-1\}'$. L'intersection de $\{p\}'$ avec $[p]_0''$ est un point non situé sur $[p-1]_0''$. Par conséquent, la variété des $\{p\}'$ est homéomorphe au produit topologique de la cellule ouverte A par la cellule ouverte $[p]_0'' - [p-1]_0''$. Par projection stéréographique de centre $[0]$, il lui correspond sur Q_{2p} une variété d'éléments $\{k\}$ qui forme bien une cellule algébrique ouverte. On voit facilement que la frontière de cette cellule est formée par les variétés indiquées dans l'énoncé du lemme.

Si $a_0 = 0$, on vérifie le lemme en remarquant que la variété des génératrices $\{p\}$ de Q_{2p} qui passent par le point $[0]$ correspond d'une façon biunivoque à la variété des génératrices $\{p-1\}$ de Q_{2p-2}.

Nous dirons que les variétés $[a_0, a_1, \cdots, a_{2i}, \cdot, \cdot\cdot]$ *sont les variétés fondamentales de* V. *En vertu du lemme, elles subdivisent* V *en cellules algébriques complexes. Elles forment donc les bases d'homologie pour les dimensions paires. Les nombres de Betti pour les dimensions impaires sont nuls, et il n'y a pas de coefficients de torsion.*

Donnons explicitement les bases d'homologie pour la variété V engendrée par les génératrices $\{3\}$ d'une quadrique Q_6.

$$
\begin{array}{ll}
[3, \cdot, \cdot, \cdot] & R_{12} = 1 \\
[2, \cdot, \cdot, \cdot] & R_{10} = 1 \\
[1, \cdot, \cdot, \cdot] & R_{8} = 1 \\
[0, \cdot, \cdot, \cdot]\ [1, 2, 3, \cdot] & R_{6} = 2 \\
[0, 2, 3, \cdot] & R_{4} = 1 \\
[0, 1, 3, \cdot] & R_{2} = 1 \\
[0, 1, 2, \cdot] & R_{0} = 1
\end{array}
$$

On reconnait que V a la même dimension et les mêmes constantes topologiques que Q_6. V et Q_6 sont effectivement homéomorphes. On connait, en effet, une correspondance biunivoque entre les points de la quadrique Q_6 et ses génératrices de l'un des deux systèmes de génératrices à 3 dimensions.[30]

Remarquons qu'on pourrait remplacer les symboles $[a_0, \cdots, a_{2i}, \cdot, \cdot\cdot]$ par des symboles légèrement différents. Soit $[p]_1$ une génératrice de Q_{2p} coupant $[p]_0$ suivant la variété $[p-1]$ de la suite (1). $[p]_0$ et $[p]_1$ sont des génératrices de systèmes différents. Considérons la suite de variétés planes:

$$(1)' \qquad [p]_1 \supset [p-1] \supset \cdots \supset [1] \supset [0].$$

Si $\{p\}$ est un élément de V, son intersection avec $[p]_1$ est nulle ou a un nombre impair de dimensions. Prenons $2i$ variétés appartenant à la suite (1)': $[a_0] \subset [a_1] \subset \cdots \subset [a_{2i-1}]$. Le symbole $[a_0, a_1, \cdots, a_{2i-1}, \cdot, \cdot\cdot]$ représentera la variété des éléments $\{p\}$ qui coupent $[a_s]$ suivant une variété à s dimensions. La variété V elle-même serait ainsi représentée par $[\cdot, \cdot, \cdots]$, où figurent seulement $p+1$ points. Nous dirons qu'un symbole est de *première* ou de *seconde* espèce, suivant qu'il contient $2i+1$ ou $2i$ nombres. Toute variété fondamentale est représentée par un symbole de première espèce et par un symbole de seconde espèce. Ces deux symboles contiennent les mêmes nombres inférieurs à p; le nombre p figure toujours dans l'un des symboles et ne figure pas dans l'autre. Par exemple, $[0, 2, 3, \cdot, \cdot]$ et $[0, 2, 3, 4, \cdot]$ représentent la même variété; de même $[1, 3, 4, \cdot, \cdot]$ et $[1, 3, \cdot, \cdot, \cdot]$ représentent la même variété. Le lemme s'énonce de la même façon quand on emploie des symboles de seconde espèce.

La dimension complexe de la variété $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ est égale à $\frac{p(p+1)}{2} - (p-a_0) - (p-a_1) - \cdots - (p-a_s)$. En effet, cette expression diminue d'une unité quand on remplace un des nombres a_j par $a_j - 1$ ou quand on remplace deux points qui suivent a_s par $p-1$ et p. D'autre part elle prend toutes les valeurs entières de $p(p+1)/2$ à 0. Remarquons qu'on peut associer à toute variété $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ la variété $[b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot]$ telle que les nombres $a_0, a_1, \cdots, a_s, b_0, b_1, \cdots, b_t$ soient, à l'ordre près, les nombres de

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[30] Ceci est lié à l'existence d'un parallélisme absolu dans l'espace elliptique réel à 7 dimensions. Voir: F. Vaney, *a*.

la suite $0, 1, \cdots, p$. Les deux variétés sont de dimensions complémentaires et les deux symboles seront dits *symboles associés.*

Démontrons le théorème suivant:

THÉORÈME. *Etant données deux variétés fondamentales de dimensions complémentaires, leur indice de Kronecker est égal à* 1 *lorsqu'elles sont représentées par deux symboles associés; sinon il est égal à zéro.*

Soient $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ et $[b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot]$ deux symboles associés, le premier étant de première espèce. Considérons les variétés planes de la suite (1) ainsi que leurs variétés conjuguées par rapport à Q_{2p}; soit $[2p - k]$ la variété conjugée de $[k]$. Nous avons alors une suite de variétés planes:

$$[2p + 1] \supset [2p] \supset \cdots \supset [p]_0 \supset [p - 1] \supset \cdots \supset [0]. \tag{3}$$

Montrons que les éléments $\{p\}$ de la variété $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ appartiennent à la variété fondamentale de Schubert définie par le symbole

$$[a_0, a_1, \cdots, a_s, n - b_t, \cdots, n - b_1, n - b_0]$$

à l'aide de $p + 1$ variétés de la suite (3); nous avons posé $n = 2p + 1$. En effet, comme un tel élément $\{p\}$ coupe $[a_i]$ suivant une variété à i dimensions, il coupe la variété conjuguée $[2p - a_i]$ suivant une variété à $\varphi(i)$ dimensions, où $\varphi(i) = p - a_i + i$. On a: $\varphi(i) - \varphi(i + 1) = a_{i+1} - a_i - 1$. La différence $\varphi(j) - \varphi(i)$ indique combien des nombres $b_0, b_1, \cdots, b_t$ sont compris entre a_j et a_i. Si dans le symbole $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ tous les nombres de a_j à a_i sont des nombres consécutifs, on a $\varphi(i) = \varphi(j)$. Soit a_i un nombre du symbole $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ tel que $a_i + 1$ soit égal à un nombre $b_{i'}$ du symbole $[b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot]$. L'élément $\{p\}$ coupe $[n - b_{i'}]$ suivant une variété à $\varphi(i)$ dimensions. Remarquons que $\varphi(i) = p - (a_i - i) = p - i'$. Il en résulte bien que $\{p\}$ fait partie de la variété fondamentale de Schubert de symbole

$$[a_0, a_1, \cdots, a_s, n - b_t, \cdots, n - b_1, n - b_0].$$

De même les éléments $\{p\}$ de la variété $[b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot]$ font partie d'une variété de Schubert de symbole $[b_0, b_1, \cdots, b_t, n - a_s, \cdots, n - a_1, n - a_0]$. Si $[b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot]$ est de seconde espèce, il faut considérer à la place de la suite (3) l'ensemble des variétés de la suite (1)′ et de leurs conjuguées par rapport à Q_{2p}. En déplaçant la génératrice $[p]_0$, nous pouvons donner à la variété $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ une position générale par rapport à la variété $[b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot]$. Ces deux variétés sont alors contenues dans deux variétés de Schubert qui ont un élément commun. On reconnait que cet élément commun appartient à Q_{2p}. Donc l'indice de Kronecker des deux variétés associées est bien égal à 1.

Soit $[c_0, c_1, \cdots, c_r, \cdot, \cdot\cdot]$ une autre variété fondamentale dont la dimension est complémentaire à celle de $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$. Nous supposons que les variétés planes qui servent à définir $[c_0, c_1, \cdots, c_r, \cdot, \cdot\cdot]$ appartiennent à une suite $[p]' \supset [p - 1]' \supset \cdots \supset [0]'$, où $[p]'$ est une génératrice qui n'a pas de point commun avec $[p]_0$. Si $r + s > p - 1$, les deux variétés

$[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ et $[c_0, c_1, \cdots, c_r, \cdot, \cdot\cdot]$ n'ont pas d'élément commun; sinon $[a_s]$ et $[c_r]'$ auraient un point commun, ce qui est impossible. Supposons $s + r \leqq p - 1$; d'où $r \leqq t$. Comme $[b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot]$ et $[c_0, c_1, \cdots, c_r, \cdot, \cdot\cdot]$ ont la même dimension, on a:

$$p - b_0 + p - b_1 + \cdots + p - b_t = p - c_0 + p - c_1 + \cdots + p - c_r.$$

Au moins un des nombres $c_0, c_1, \cdots, c_r$ est donc inférieur au nombre de même rang de la suite $b_0, b_1, \cdots, b_t$; soit $c_i < b_i$. Alors les variétés $[n - b_i]$ et $[c_i]'$ n'ont pas de point commun. Donc les variétés $[c_0, c_1, \cdots, c_r, \cdot, \cdot\cdot]$ et $[a_0, a_1, \cdots, a_s, n - b_t, \cdots, n - b_1, n - b_0]$ n'ont pas d'élément commun.

Du théorème ainsi démontré il résulte que tout cycle Γ_{2k} de la variété V s'exprime par une homologie de la forme:

$$\Gamma_{2k} \sim \Sigma\, (b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot)\, [a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot],$$

où $(b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot]$ est l'indice de Kronecker de Γ_{2k} avec la variété fondamentale associée à $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$.

16. **L'espace des variétés planes à p dimensions qui appartiennent à un complexe linéaire non dégénéré d'un espace projectif à $2p + 1$ dimensions complexes.** Soit C un complexe linéaire de droites défini dans l'espace projectif complexe $[2p + 1]$. Nous supposons que C est non dégénéré. Les droites de C qui passent par un point M remplissent un hyperplan $[2p]$, qui est appelé hyperplan conjugué de M. A une variété $[i]$ est associée une variété conjuguée $[2p - i]$; c'est la variété commune à tous les hyperplans conjugués des points de $[i]$. Si $i < p$, il y a des variétés $[i]$ qui sont contenues dans leurs variétés conjuguées. Nous disons qu'une telle *variété* $[i]$ *appartient au complexe*; toute droite de $[i]$ est alors une droite du complexe. Une variété $[p]$ qui appartient au complexe est confondue avec sa conjuguée. Soit V la variété des $[p]$ qui appartiennent au complexe. Les propriétés de V ressemblent à celles de la variété des génératrices à p dimensions d'une quadrique Q_{2p}.[31] Désignons par $\{p\}$ un élément quelconque de V et soit $[p]_0$ un élément particulier de V. Nous allons démontrer le lemme suivant:

LEMME I. *La variété V devient une cellule algébrique ouverte lorsqu'on en enlève tous les $\{p\}$ qui rencontrent $[p]_0$.*

La démonstration se fait de nouveau par induction. Soit O un point de $[p]_0$ et soit $[2p]_0$ l'hyperplan conjugué de O. $[2p]_0$ contient $[p]_0$. Tout $\{p\}$ qui ne rencontre pas $[p]_0$ coupe $[2p]_0$ suivant une variété $\{p - 1\}$ qui ne rencontre pas $[p]_0$. Soit $\{p + 1\}$ la variété conjuguée de $\{p - 1\}$. Les éléments $\{p\}$ qui passent par $\{p - 1\}$ sont les variétés planes à p dimensions qui passent par $\{p - 1\}$ et qui sont contenues dans $\{p + 1\}$. Considérons une variété $[p - 1]_0$ contenue dans $[p]_0$ et ne passant pas par O. La variété $[p + 1]_0$, conjuguée de

[31] Pour les propriétés classiques d'un complexe linéaire, voir C. Segre, *a*.

$[p-1]_0$, coupe $[2p]_0$ suivant $[p]_0$. $\{p+1\}$ et $[p+1]_0$ se coupent suivant une droite passant par O. Tout élément $\{p\}$ passant par $\{p-1\}$ coupe $[p+1]_0$ suivant un point et correspond ainsi d'une façon biunivoque à une variété $[p]'$ passant par $[p-1]_0$ et contenue dans $[p+1]_0$. Les éléments $\{p\}$ qui passent par $[p-1\}$ engendrent une cellule ouverte lorsqu'on enlève l'élément qui passe par O. Le lemme sera donc démontré quand on aura montré que les variétés $\{p-1\}$ qui ne rencontrent pas $[p]_0$ forment une cellule ouverte.

Considérons dans $[2p]_0$ une variété $[2p-1]_0$ qui coupe $[p]_0$ suivant $[p-1]_0$. Les droites du complexe C qui sont contenues dans $[2p-1]_0$ engendrent un complexe linéaire non dégénéré C'. Par projection centrale de centre O, toute variété $\{p-1\}$ dans $[2p]_0$ qui appartient au complexe C et qui ne rencontre pas $[p]_0$ se projette sur $[2p-1]_0$ suivant une variété $\{p-1\}'$ qui appartient au complexe C' et qui ne rencontre pas $[p-1]_0$. Par hypothèse, les variétés $\{p-1\}'$ engendrent une cellule algébrique ouverte. Les variétés $\{p-1\}$ qui se projettent suivant une variété $\{p-1\}'$ donnée et qui ne passent pas par O engendrent une cellule ouverte. On peut les mettre en correspondance biunivoque avec les variétés $[2p-1]$ qui sont déterminées par $\{p-1\}$ et $[p-1]_0$. Donc les $\{p-1\}$ qui ne rencontrent pas $[p]_0$ engendrent bien une cellule ouverte.

Le lemme se démontre directement pour $p=1$ en appliquant les réflexions précédentes; il est donc prouvé quel que soit p.

Considérons dans $[p]_0$ une suite de variétés planes:

$$[p]_0 \supset [p-1]_0 \supset \cdots \supset [1]_0 \supset [0]_0.$$

Le symbole $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$, où $0 \leqq a_0 < a_1 < \cdots < a_s \leqq p$, représentera la variété des éléments $\{p\}$ de V qui coupent $[a_i]_0$ suivant une variété à i dimensions. On a ajouté $p-s$ points à la suite de a_s, pour indiquer que l'élément générateur est à p dimensions.

LEMME II. *La variété* $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ *devient une cellule algébrique ouverte lorsqu'on en enlève l'ensemble des variétés représentées par les symboles qu'on obtient en diminuant d'une unité un des nombres du symbole donné ou en écrivant le nombre* p *à la suite de* a_s, *si* $a_s < p$. *Tout symbole qui n'a pas de sens est remplacé par zéro.*

Nous raisonnons de nouveau par induction par rapport à p. Le lemme se vérifie immédiatement pour $p=1$. Soit $[2p]'$ une variété plane qui coupe $[a_0]_0$ suivant $[a_0-1]_0$ et les variétés $[a_0+1]_0, [a_0+2]_0, \cdots, [p]_0$ suivant des variétés $[a_0]', [a_0+1]', \cdots, [p-1]'$. Considérons un élément général $\{p\}$ de la variété $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$. Il est déterminé par un point M de $[a_0]_0$ et une variété $\{p-1\}$ contenue dans $[2p]'$. Nous supposons que M n'appartient pas à $[a_0-1]_0$. L'hyperplan conjugué de M coupe $[2p]'$ suivant une variété $[2p-1]'$. Les droites du complexe C qui sont contenues dans $[2p-1]'$ engendrent un complexe linéaire C'. La variété $\{p-1\}$ appartient au complexe C' et engendre, pour un point M donné, la variété de symbole

$$[a_1-1, a_2-1, \cdots, a_s-1, \cdot, \cdot\cdot]'$$

définie à l'aide des variétés $[a_1 - 1]', [a_2 - 1]', \cdots, [a_s - 1]'$. Par hypothèse on peut appliquer le lemme à la variété $[a_1 - 1, a_2 - 1, \cdots, a_s - 1, \cdot, \cdot\cdot]'$. Faisons décrire à M la cellule ouverte $[a_0]_0 - [a_0 - 1]_0$. A deux points M_1 et M_2 correspondent deux hyperplans conjugués qui coupent $[2p]'$ suivant des variétés $[2p-1]'_1$ et $[2p-1]'_2$. Celles-ci ne contiennent pas le pôle I de l'hyperplan $[2p]'$. Les variétés $\{p-1\}$ dans $[2p-1]'_1$ et $[2p-1]'_2$ peuvent être mises en correspondance biunivoque par projection centrale de centre I. La variété $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ dont on a enlevé les éléments qui rencontrent $[a_0 - 1]_0$ est donc le produit topologique de $[a_0]_0 - [a_0 - 1]_0$ par la variété $[a_1 - 1, a_2 - 1, \cdots, a_s - 1, \cdot, \cdot\cdot]'$. Le lemme résulte de cette propriété.

Nous dirons encore que les variétés $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ *sont les variétés fondamentales de* V. *Elles subdivisent* V *en cellules algébriques et forment par suite les bases d'homologie.*

La dimension complexe de V est $(p+1)(p+2)/2$. En raisonnant comme au §15, on trouve que la dimension complexe de $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$ est

$$\frac{(p+1)(p+2)}{2} - (p+1-a_0) - (p+1-a_1) - \cdots - (p+1-a_s).$$

Soient $b_0, b_1, \cdots, b_t$ les nombres de la suite $0, 1, \cdots, p$ qui ne figurent pas parmi la suite $a_0, a_1, \cdots, a_s$. La variété $[b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot]$ sera dite *associée* à $[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot]$. Deux variétés associées sont de dimensions complémentaires. On montre comme au §15 que *l'indice de Kronecker de deux variétés associées est égal à* 1. *Si deux variétés de dimensions complémentaires ne sont pas des variétés associées, leur indice de Kronecker est nul.* Par conséquent tout cycle Γ_{2k} s'exprime par une homologie de la forme:

$$\Gamma_{2k} \sim \Sigma(b_0, b_1, \cdots, b_t, \cdot, \cdot\cdot)[a_0, a_1, \cdots, a_s, \cdot, \cdot\cdot],$$

les notations étant les mêmes qu'au §15.

Il y a une analogie frappante entre la variété des génératrices $\{p+1\}$ d'une quadrique Q_{2p+2} et la variété des éléments $\{p\}$ appartenant à un complexe linéaire de l'espace $[2p+1]$. Dans les deux cas, toute variété fondamentale est définie par un ensemble d'entiers $(a_0, a_1, \cdots, a_s)$ tels que $0 \leqq a_0 < a_1 < \cdots < a_s \leqq p$; le nombre $p+1$ qui peut figurer dans les symboles des variétés fondamentales du premier cas ne joue pas un rôle essentiel. Les invariants topologiques ordinaires (nombres de Betti, coefficients de torsion, groupe de Poincaré) sont les mêmes. Cependant les deux variétés ne sont pas homéomorphes. Soit, en effet, le cas $p = 1$. La variété des génératrices $\{2\}$ d'une quadrique Q_4 est homéomorphe à l'espace projectif [3]. La variété des droites d'un complexe linéaire de [3] est homéomorphe à une quadrique Q_3. On voit facilement que les variétés [3] et Q_3 ne sont pas homéomorphes. Il suffit de considérer dans les deux cas l'intersection avec lui-même du cycle de base à 2 dimensions complexes.

17. **Application de la méthode des invariants intégraux.** Il est intéressant de retrouver certains des résultats précédents par la méthode des invariants intégraux[32] du §5.

Considérons la quadrique Q_{2p} d'équation:

$$x_0 y_0 + x_1 y_1 + \cdots + x_p y_p = 0. \tag{1}$$

Soit $\{p\}_0$ la génératrice $x_0 = x_1 = \cdots = x_p = 0$. Les génératrices $\{p\}$ voisines de $\{p\}_0$ sont représentées par:

$$x_i = \sum_i a_{ij}\, y_i, \qquad a_{ij} + a_{ji} = 0.$$

La variété V des génératrices $\{p\}$ de même système que $\{p\}_0$ est transformée transitivement par le groupe projectif G qui laisse invariantes la quadrique (1) et les relations $y_i = \bar{x}_i$. G est équivalent au groupe orthogonal à paramètres réels et à $2p + 2$ variables. Le groupe d'isotropie g relatif à l'élément-origine $\{p\}_0$ est défini par:

$$\begin{aligned} (x') &= (a)\,(x) \\ (y') &= (\bar{a})\,(y) \end{aligned} \qquad (a)(\bar{a})^* = 1 \tag{g}$$

V peut être considéré comme l'espace riemannien symétrique défini par G et g, la symétrie par rapport à l'élément $\{p\}_0$ étant définie par:

$$(x') = -(x), \qquad (y') = (y).$$

Le groupe linéaire d'isotropie γ opère sur des variables complexes ω_{ij} et sur les variables complexes conjuguées $\bar{\omega}_{ij}$. On a $\omega_{ij} + \omega_{ji} = 0$. Les ω_{ij} se transforment comme les formes extérieures $[x_i\, x_j]$, en supposant que les variables $x_0, x_1, \cdots, x_p$ subissent les transformations du groupe unitaire $(x') = (a)\,(x)$, où $(a)\,(\bar{a})^* = 1$. Aux transformations unimodulaires de ce dernier groupe correspond un groupe linéaire γ'' opérant sur les ω_{ij}. On est de nouveau conduit à décomposer en groupes irréductibles le groupe γ_s'' qui opère sur les formes extérieures $[\omega_{i_1 j_1} \cdots \omega_{i_s j_s}]$. En répétant les raisonnements du §7, on obtient facilement les variables dominantes de ces groupes irréductibles. Considérons le tableau des variables ω_{ij} qui s'obtient en gardant dans la matrice (ω_{ij}) seulement les éléments placés au-dessus de la diagonale principale. Etant donné un ensemble d'entiers $l_0, l_1, \cdots, l_k$ tels que $p \geqq l_0 > l_1 > \cdots > l_k > 0$ et tels que $l_0 + l_1 + \cdots + l_k = s$, prenons les l_0 premiers éléments de la première ligne du tableau des ω_{ij}, puis les l_1 premiers éléments de la seconde ligne, etc. Le produit extérieur de tous ces éléments, $[\omega_{01}\, \omega_{02} \cdots \omega_{0l_0}\, \omega_{12}\, \omega_{13} \cdots]$, est une des variables dominantes cherchées. Toute variable dominante s'obtient ainsi. Donc:

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[32] Pour ce qui concerne les quadriques complexes, voir E. Cartan, *g*.

THÉORÈME. *Le nombre d'invariants intégraux de degré* $2s$, *ou le nombre de Betti* R_{2s}, *est égal au nombre d'ensembles d'entiers* $(l_0, l_1, \cdots, l_k)$ *tels que*:

$$p \geqq l_0 > l_1 > \cdots > l_k \geqq 0, \qquad l_0 + l_1 + \cdots + l_k = s.$$

Considérons de même le complexe linéaire défini par l'équation:

$$[x_0\, y_0] + [x_1\, y_1] + \cdots + [x_p\, y_p] = 0. \tag{2}$$

La variété $x_0 = x_1 = \cdots = x_p = 0$ est une variété $\{p\}_0$ appartenant au complexe linéaire. Soit $\{p\}$ une variété quelconque appartenant au complexe linéaire. Les variétés $\{p\}$ voisines de $\{p\}_0$ sont représentées par:

$$x_i = \sum_j a_{ij}\, y_j, \qquad a_{ij} = a_{ji}.$$

La variété W engendrée par les $\{p\}$ est transformée transitivement par le groupe linéaire G qui laisse invariants le premier membre de l'équation (2) ainsi que la forme d'Hermite $\sum_i x_i \bar{x}_i + \sum_i y_i \bar{y}_i$. Le sous-groupe g qui laisse invariant $\{p\}_0$ est de nouveau défini par:

$$(x') = (a)(x), \qquad (y') = (\bar{a})(y), \qquad (a)(\bar{a})^* = 1.$$

Le groupe linéaire d'isotropie γ opère sur des variables complexes ω_{ij} et $\bar{\omega}_{ij}$ telles que $\omega_{ij} = \omega_{ji}$. ω_{ij} est transformé comme le produit $x_i\, x_j$ lorsque les variables x_i subissent les transformations du groupe $(x') = (a)(x)$, où $(a)(\bar{a})^* = 1$. On peut encore considérer le tableau des ω_{ij} obtenu en supprimant dans la matrice (ω_{ij}) les éléments placés *au-dessous* de la diagonale principale. On trouvera par la même règle les variables dominantes appartenant aux groupes γ_s'' et on a le même théorème que dans le cas précédent, les entiers $(l_0, l_1, \cdots, l_k)$ satisfaisant maintenant aux conditions: $p + 1 \geqq l_0 > l_1 > \cdots > l_k \geqq 0$, $l_0 + l_1 + \cdots + l_k = s$.

V. Les propriétés topologiques d'une classe de variétés ayant comme élément générateur une figure formée de plusieurs variétés planes

18. **Définitions et propriétés générales.** Les raisonnements des paragraphes précédents peuvent s'appliquer à une classe étendue de variétés algébriques qu'on peut considérer comme des généralisations des variétés de Grassmann. L'exemple le plus simple est fourni par la variété des éléments linéaires de l'espace projectif complexe $[n]$, en appelant élément linéaire l'ensemble d'un point et d'une droite passant par ce point. On peut considérer plus généralement l'ensemble d'une variété $[\alpha]$ et d'une variété $[\beta]$, où l'on suppose $\alpha < \beta$ et $[\alpha] \subset [\beta]$. Cette figure peut être prise comme élément générateur d'une variété et sera représentée par le symbole $\{\alpha, \beta\}$. Ainsi un élément linéaire est représenté par $\{0, 1\}$. On peut encore considérer des variétés ayant comme élément générateur la figure formée par $k + 1$ variétés $[\alpha_0], [\alpha_1], \cdots, [\alpha_k]$ telles que:

$$\alpha_0 < \alpha_1 < \cdots < \alpha_k; \qquad [\alpha_0] \subset [\alpha_1] \subset \cdots \subset [\alpha_k] \subset [n].$$

Un tel élément générateur sera représenté par $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}$. Remarquons que chaque élément $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}$ définit aussi une variété fondamentale de Schubert.

Soit V la variété de tous les éléments $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}$ contenus dans $[n]$, les nombres $\alpha_0, \alpha_1, \cdots, \alpha_k$ étant des nombres donnés. D'après le théorème fondamental de la géométrie projective, il existe une transformation homographique de $[n]$ qui transforme un élément arbitraire $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}_1$ en un autre élément arbitraire $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}_2$. La variété V est donc transformée transitivement par le groupe projectif de $[n]$. En raisonnant comme dans le cas des variétés de Grassmann, on montre que V est aussi transformée transitivement par le groupe hermitien elliptique. V définit donc encore un espace homogène clos ayant pour groupe de structure le groupe hermitien elliptique. Mais cet espace homogène n'est pas symétrique.

La dimension complexe de V est égale à

$$(\alpha_0 + 1)(\alpha_1 - \alpha_0) + (\alpha_1 + 1)(\alpha_2 - \alpha_1) + \cdots + (\alpha_k + 1)(n - \alpha_k).$$

La caractéristique d'Euler-Poincaré de V s'obtient encore en appliquant le théorème de M. S. Lefschetz. Une homographie générale de $[n]$ laisse fixes les $n + 1$ sommets d'un simplexe non dégénéré. Le nombre d'éléments $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}$ qui sont fixes par l'homographie est égal à

$$\binom{n+1}{\alpha_k+1}\binom{\alpha_k+1}{\alpha_{k+1}+1}\cdots\binom{\alpha_1+1}{\alpha_0+1}.$$

Ce nombre est égal à la caractéristique d'Euler-Poincaré, car en vertu du théorème démontré à la fin du §7, chaque élément fixe doit être compté avec une multiplicité égale à 1.

19. **La variété des éléments $\{\alpha, \beta\}$ de $[n]$.** La variété de tous les éléments (α, β) de $[n]$ est une variété algébrique V contenue dans le produit topologique de la variété des éléments $\{\alpha\}$ par la variété des éléments $\{\beta\}$ de $[n]$. Considérons de nouveau une suite de variétés planes $[n-1], [n-2], \cdots, [1], [0]$ telles que:

$$[n] \supset [n-1] \supset [n-2] \supset \cdots \supset [1] \supset [0]. \tag{1}$$

Elles déterminent les variétés fondamentales $[a_0, a_1, \cdots, a_\alpha]$ engendrées par des éléments $\{\alpha\}$ et les variétés fondamentales $[b_0, b_1, \cdots, b_\beta]$ engendrées par des éléments $\{\beta\}$. Le symbole $\begin{bmatrix} a_0, a_1, \cdots, a_\alpha \\ b_0, b_1, \cdots, b_\alpha, \cdots, b_\beta \end{bmatrix}$ représentera la variété des éléments $\{\alpha, \beta\}$ définis par l'ensemble d'un $\{\alpha\}$ appartenant à $[a_0, a_1, \cdots, a_\alpha]$ et d'un $\{\beta\}$ appartenant à $[b_0, b_1, \cdots, b_\beta]$. Pour que $\{\alpha\}$ et $\{\beta\}$ puissent être des éléments généraux des variétés $[a_0, a_1, \cdots, a_\alpha]$ et $[b_0, b_1, \cdots, b_\beta]$, il faut et il suffit que tout nombre a_i soit égal à un nombre b_j dont l'indice j est supérieur ou égal à i. Lorsque cette condition est satisfaite, le symbole considéré sera dit irréductible et représentera une variété irréductible que nous appellerons

variété fondamentale. Lorsque cette condition n'est pas satisfaite, le symbole est réductible et représente, en général, la somme de plusieurs variétés fondamentales. Soit, en effet, a_i le premier nombre de la première ligne du symbole qui ne figure pas parmi les nombres b_j de la deuxième ligne. Supposons a_i compris entre b_j et b_{j+1} et soit $\{\alpha, \beta\}$ un élément de la variété défini par le symbole donné. Si l'intersection de $[a_i]$ et de $\{\alpha\}$ ne se trouve pas entièrement dans $[b_j]$, la variété $\{\beta\}$ coupera $[a_i]$ suivant une variété $[j+1]$. Ceci montre qu'on peut remplacer le symbole donné par la somme des deux symboles obtenus en remplaçant soit a_i par b_j, soit b_{j+1} par a_i. Cependant lorsque $b_j = a_{i-1}$, il faudra remplacer a_i par b_j et diminuer d'une unité un ou plusieurs des nombres qui précèdent a_i. Ainsi $\begin{bmatrix} 2, 3 \\ 0, 2, 4 \end{bmatrix}$ doit être remplacé par $\begin{bmatrix} 2, 3 \\ 0, 2, 3 \end{bmatrix} + \begin{bmatrix} 1, 2 \\ 0, 2, 4 \end{bmatrix}$. Si les symboles obtenus ne sont pas irréductibles, on applique la même opération jusqu'à ce qu'on arrive à une somme de symboles irréductibles.

Nous allons montrer que les variétés fondamentales fournissent une subdivision de V en cellules algébriques. Pour éviter des longueurs, nous supposerons que V est la variété des éléments $\{1, 6\}$ de l'espace $[n]$; on reconnaîtra immédiatement que le raisonnement est général. Soit $\begin{bmatrix} a_0, a_1 \\ b_0, a_0, b_2, b_3, a_1, b_5, b_6 \end{bmatrix}$ le symbole d'une variété fondamentale F. Nous choisissons dans $[n]$ une variété $[n-2]'$ qui ne rencontre pas les droites générales de $[a_0, a_1]$. Il faudra donc que $[n-2]'$ coupe $[a_0]$ suivant $[a_0 - 1]$ et $[a_1]$ suivant une variété $[a_1 - 2]'$ qui n'appartient pas à la suite (1) mais qui est contenue dans $[a_1 - 1]$. Alors $[n-2]'$ contient $[b_0]$ et coupe les variétés $[b_2]$, $[b_3]$, $[b_5]$, $[b_6]$ suivant des variétés $[b_2 - 1]'$, $[b_3 - 1]'$, $[b_5 - 2]'$, $[b_6 - 2]'$. Soit $\{1, 6\}$ un élément de la variété F. Il est composé d'une droite $\{1\}$ et d'une variété $\{6\}$. Nous supposons que la droite $\{1\}$ n'appartient pas à $[a_0, a_1 - 1] + [a_0 - 1, a_1]$. Il s'en suit que $\{1\}$ ne rencontre pas la variété $[n-2]'$. L'intersection de $\{6\}$ et de $[n-2]'$ sera une variété $\{4\}$ qui appartient à la variété $[b_0, b_2 - 1, b_3 - 1, b_5 - 2, b_6 - 2]'$ définie à l'aide de $[b_0]$, $[b_2 - 1]'$, $[b_3 - 1]'$, $[b_5 - 2]'$, $[b_6 - 2]'$. Réciproquement, l'ensemble d'une droite $\{1\}$ appartenant à $[a_0, a_1] - [a_0, a_1 - 1] - [a_0 - 1, a_1]$ et d'un élément $\{4\}$ de la variété $[b_0, b_2 - 1, b_3 - 1, b_5 - 2, b_6 - 2]'$ détermine un élément $\{1, 6\}$ appartenant à la variété F. Donc si l'on enlève de F les éléments $\{1, 6\}$ dont la droite $\{1\}$ appartient à $[a_0, a_1 - 1] + [a_0 - 1, a_1]$, on obtient une variété qui est le produit topologique de la cellule ouverte $[a_0, a_1] - [a_0, a_1 - 1] - [a_0 - 1, a_1]$ et de la variété $[b_0, b_2 - 1, b_3 - 1, b_5 - 2, b_6 - 2]'$. Pour faire de cette dernière variété une cellule ouverte, il faut en enlever des variétés ayant pour symboles $[b_0 - 1, b_2 - 1, b_3 - 1, b_5 - 2, b_6 - 2]'$, $[b_0, b_2 - 2, \cdots]'$, $\cdots$, $[b_0, b_2 - 1, b_3 - 1, b_5 - 3, b_6 - 2]'$, $\cdots$. Les variétés planes qui servent à définir ces symboles sont les intersections des variétés de la suite (1) avec la variété $[n-2]'$. Les éléments $\{6\}$ qu'on a enlevés de cette façon sont les éléments des variétés représentées par les symboles obtenus à partir de $[b_0, a_0, b_2, b_3, a_1, b_5, b_6]$ en diminuant d'une unité un des nombres b_0, b_2, b_3, b_5, b_6. Une circonstance exceptionnelle se présente lorsque $b_2 - 1 = a_0$ ou $b_5 - 1 = a_1$.

Si $b_2 - 2 = b_0$, on n'a pas à considérer le symbole obtenu en remplaçant b_2 par $b_2 - 1$. Mais si $b_2 - 1 > b_0 + 1$, il faut remplacer a_0 et b_2 par $a_0 - 1$ et a_0. Ce qui précède permet d'énoncer, pour tous les cas, la règle suivante:

La variété fondamentale F de symbole $\begin{bmatrix} a_0, a_1 \\ b_0, a_0, b_2, b_3, a_1, b_5, b_6 \end{bmatrix}$ *devient une cellule algébrique ouverte lorsqu'on en enlève les variétés représentées par les symboles qui se déduisent du symbole de F en dimiunant d'une unité un des nombres de la première ou de la deuxième ligne. On ne garde que les symboles qui ont un sens et on les décompose en symboles irréductibles.*

On reconnait que notre raisonnement est tout à fait général; la même règle s'applique donc dans le cas des variétés fondamentales engendrées par un élément $\{\alpha, \beta\}$ quelconque. Par conséquent:

Les variétés fondamentales $\begin{bmatrix} a_0, a_1, \cdots, a_\alpha \\ b_0, b_1, \cdots\cdots, b_\beta \end{bmatrix}$ *subdivisent la variété V en cellules algébriques et fournissent les bases d'homologie. Il n'y a pas de coefficients de torsion et les nombres de Betti* R_{2s+1} *sont nuls.*

M. E. Cartan a bien voulu m'indiquer une autre manière de représenter les variétés fondamentales et de définir les cellules algébriques correspondantes. D'après ce qui précède, on voit que la dimension complexe de la variété $\begin{bmatrix} a_0, a_1, \cdots, a_\alpha \\ b_0, b_1, \cdots\cdots, b_\beta \end{bmatrix}$ est égale à $\sum_i (a_i - i) + \sum_j (b_j - j)$, où $i = 0, 1, \cdots, \alpha$, tandis que j ne prend que les valeurs des indices des nombres b_λ qui ne figurent pas dans la première ligne du symbole. La variété est complètement définie par l'ensemble des nombres $(a_i - i)$ et $(b_j - j)$, c'est-à-dire par $\beta + 1$ nombres entiers dont les $\alpha + 1$ premiers ne dépassent pas $n - \alpha$ et sont rangés par ordre non décroissant, tandis que les $\beta - \alpha$ nombres suivants ne dépassent pas $n - \beta$ et sont rangés par ordre non décroissant. Ainsi la variété $\begin{bmatrix} 2, 4 \\ 1, 2, 3, 4, 6 \end{bmatrix}$ serait définie par le système d'entiers $(2, 3 \mid 1, 1, 2)$; sa dimension est égale à 9. La cellule ouverte correspondante est définie de la façon suivante: Considérons les sommets $A_0, A_1, \cdots, A_n$ d'un simplexe de référence, les $p + 1$ premiers points définissant la variété $[p]$ de la suite (1). Formons le tableau suivant:

$A_0A_1A_2$	$2 + 1$ points
$A_0A_1A_3A_4$	$3 + 1$ points
A_0A_1	$1 + 1$ points
A_0A_3	$1 + 1$ points
$A_0A_5A_6$	$2 + 1$ points.

Dans chaque ligne nous évitons d'écrire les derniers points des lignes précédentes. La droite de l'élément $\{1, 4\}$ qui engendre la cellule ouverte est définie par les points $A_2 + (A_0A_1)$ et $A_4 + (A_0A_1A_3)$. La variété $\{4\}$ correspondante est définie par les points précédents plus les trois points $A_1 + (A_0)$, $A_3 + (A_0)$ et $A_6 + (A_0A_5)$. Ici $(A_iA_jA_k)$ représente une combinaison linéaire à coefficients

arbitraires des points A_i, A_j, A_k. Les derniers indices des différentes lignes du tableau sont les nombres qui composent le symbole $\begin{bmatrix} 2, 4 \\ 1, 2, 3, 4, 6 \end{bmatrix}$.

Il est facile d'obtenir *l'indice de Kronecker de deux variétés fondamentales de dimensions complémentaires.* Soient

$$\begin{bmatrix} a_0, a_1, \cdots, a_\alpha \\ b_0, b_1, \cdots, b_\alpha, \cdots, b_\beta \end{bmatrix} \text{ et } \begin{bmatrix} a'_0, a'_1, \cdots, a'_\alpha \\ b'_0, b'_1, \cdots, b'_\alpha, \cdots, b'_\beta \end{bmatrix}$$

les symboles irréductibles des deux variétés considérées. Donnons à la deuxième variété une position générale par rapport à la première; c'est-à-dire nous supposons que les variétés $[b'_i]$ qui servent à définir la deuxième variété fondamentale ont une position générale par rapport aux variétés $[b_j]$ qui servent à définir la première. Pour qu'il y ait un élément $\{\alpha, \beta\}$ commun aux deux variétés il faut qu'on ait, d'après un raisonnement déjà fait à propos des variétés de Grassmann:

$$a_0 + a'_\alpha \geqq n, \qquad a_1 + a'_{\alpha-1} \geqq n, \cdots, a_\alpha + a'_0 \geqq n$$
$$b_0 + b'_\beta \geqq n, \qquad b_1 + b'_{\beta-1} \geqq n, \cdots, b_\beta + b'_0 \geqq n.$$

En particulier, si la deuxième variété fondamentale a pour symbole

$$\begin{bmatrix} n - a_\alpha, \cdots, n - a_0 \\ n - b_\beta, \cdots\cdots\cdots, n - b_1, n - b_0 \end{bmatrix},$$

il y a un élément et un seul commun aux deux variétés. Les nombres a'_i et b'_i devront être supérieurs ou égaux aux nombres correspondants dans le symbole $\begin{bmatrix} n - a_\alpha, \cdots, n - a_0 \\ n - b_\beta, \cdots\cdots\cdots, n - b_1, n - b_0 \end{bmatrix}$. Or quand on a deux variétés fondamentales F_1 et F_2, définies à l'aide des variétés planes de la suite (1), F_1 est contenu dans F_2 lorsque les nombres du symbole irréductible de F_2 sont supérieurs ou égaux aux nombres correspondants du symbole irréductible de F_1. Dans ces conditions, les deux variétés F_1 et F_2 ne peuvent avoir des dimensions égales que lorsque leurs symboles irréductibles sont identiques. Donc le symbole d'une variété fondamentale ayant un indice de Kronecker non nul par rapport à $\begin{bmatrix} a_0, \cdots, a_\alpha \\ b_0, \cdots\cdots, b_\beta \end{bmatrix}$ ne peut être que $\begin{bmatrix} n - a_\alpha, \cdots, n - a_0 \\ n - b_\beta, \cdots\cdots\cdots, n - b_0 \end{bmatrix}$.

Montrons que:

$$\begin{bmatrix} a_0, \cdots, a_\alpha \\ b_0, \cdots\cdots, b_\beta \end{bmatrix} \cdot \begin{bmatrix} n - a_\alpha, \cdots, n - a_0 \\ n - b_\beta, \cdots\cdots\cdots, n - b_0 \end{bmatrix} = 1.$$

Pour cela nous considérons seulement l'exemple des variétés:

$$\begin{bmatrix} a_0, a_1 \\ b_0, a_0, b_2, b_3, a_1, b_5, b_6 \end{bmatrix} \text{ et } \begin{bmatrix} n - a_1, n - a_0 \\ n - b_6, n - b_5, n - a_1, n - b_3, n - b_2, n - a_0, n - b_0 \end{bmatrix}.$$

On verra que le raisonnement est général. Nous avons considéré une variété $[n - 2]'$ coupant $[a_0]$ suivant $[a_0 - 1]$ et $[a_1]$ suivant une variété $[a_1 - 2]'$ con-

tenue dans $[a_1 - 1]$. La variété des éléments $\{1, 6\}$ de $\begin{bmatrix} a_0, a_1 \\ b_0, a_0, b_2, b_3, a_1, b_5, b_6 \end{bmatrix}$ dont la droite ne rencontre pas $[n-2]'$ est alors le produit topologique de $[a_0, a_1] - [a_0 - 1, a_1] - [a_0, a_1 - 1]$ par une variété $[b_0, b_2 - 1, b_3 - 1, b_5 - 2, b_6 - 2]'$ définie dans $[n-2]'$. Nous pouvons aussi considérer une variété $[n-2]''$ coupant $[n - a_1]$ suivant $[n - a_1 - 1]$ et $[n - a_0]$ suivant une variété $[n - a_0 - 2]''$ contenue dans $[n - a_0 - 1]$. Alors la variété des éléments $\{1, 6\}$ de $\begin{bmatrix} n - a_1, n - a_0 \\ n - b_6, n - b_5, n - a_1, n - b_3, n - b_2, n - a_0, n - b_0 \end{bmatrix}$ dont la droite ne rencontre pas $[n-2]''$ est le produit topologique de

$$[n - a_1, n - a_0] - [n - a_1 - 1, n - a_0] - [n - a_1, n - a_0 - 1]$$

par une variété $[n - b_6, n - b_5, n - b_3 - 1, n - b_2 - 1, n - b_0 - 2]''$ définie dans $[n-2]''$. Par une transformation homographique, nous pouvons amener $\begin{bmatrix} n - a_1, n - a_0 \\ n - b_6, n - b_5, n - a_1, n - b_3, n - b_2, n - a_0, n - b_0 \end{bmatrix}$ en une position générale par rapport à $\begin{bmatrix} a_0, a_1 \\ b_0, a_0, b_2, b_3, a_1, b_5, b_6 \end{bmatrix}$ de telle façon que $[n-2]''$ vienne en coïncidence avec $[n-2]'$. Nous indiquons par un indice inférieur égal à 1 tout ce qui se rapporte à la variété qu'on a déplacée. Nous pouvons supposer que les variétés $[n - a_1 - 1]$ et $[n - a_0 - 2]''$ viennent en coïncidence avec des variétés $[n - a_1 - 1]_1$ et $[n - a_0 - 2]''_1$ de $[n-2]'$ telles que $[a_0 - 1]$ et $[n - a_0 - 2]''_1$ ne se rencontrent pas et telles que $[a_1 - 2]'$ et $[n - a_1 - 1]_1$ ne se rencontrent pas. Alors les deux variétés fondamentales considérées ne peuvent avoir aucun élément d'intersection $\{1, 6\}$ dont la droite $\{1\}$ rencontre $[n-2]'$. En ce qui concerne l'indice de Kronecker des deux variétés, il suffit donc d'en considérer les parties engendrées par des éléments $\{1, 6\}$ dont la droite ne rencontre pas $[n-2]'$. Les deux variétés dont il faut chercher l'indice de Kronecker sont donc le produit topologique de $[a_0, a_1] - [a_0 - 1, a_1] - [a_0, a_1 - 1]$ par $[b_0, b_2 - 1, b_3 - 1, b_5 - 2, b_6 - 2]'$ et le produit topologique de

$$[n - a_1, n - a_0]_1 - [n - a_1 - 1, n - a_0]_1 - [n - a_1, n - a_0 - 1]_1$$

par $\quad [n - b_6, n - b_5, n - b_3 - 1, n - b_2 - 1, n - b_0 - 2]''_1.$

La variété totale des éléments $\{1, 6\}$ dont la droite ne rencontre pas $[n-2]'$ est le produit topologique de la variété des droites qui ne rencontrent pas $[n-2]'$ par la variété des $\{4\}$ de $[n-2]'$. De tout cela il résulte que l'indice de Kronecker cherché est égal à $+1$, car l'indice de Kronecker de $[a_0, a_1]$ avec $[n - a_1, n - a_0]_1$ est égal à $+1$ et l'indice de Kronecker de $[b_0, b_2 - 1, b_3 - 1, b_5 - 2, b_6 - 2]'$ avec $[n - b_6, n - b_5, n - b_3 - 1, n - b_2 - 1, n - b_0 - 2]''_1$ est égal à $+1$.

Soit Γ_{2s} un cycle sur V de dimension $2s$. On aura l'homologie:

$$\Gamma_{2s} \sim \Sigma \begin{pmatrix} n - a_\alpha, \cdots, n - a_0 \\ n - b_\beta, \cdots\cdots\cdots, n - b_0 \end{pmatrix} \begin{bmatrix} a_0, \cdots, a_\alpha \\ b_0, \cdots\cdots, b_\beta \end{bmatrix},$$

le facteur $\begin{pmatrix} n-a_\alpha, \cdots, n-a_0 \\ n-b_\beta, \cdots\cdots, n-b_0 \end{pmatrix}$ étant l'indice de Kronecker de Γ_{2s} et de $\begin{bmatrix} n-a_\alpha, \cdots, n-a_0 \\ n-b_\beta, \cdots\cdots, n-b_0 \end{bmatrix}$. *Ceci résout en même temps le problème des caractéristiques de Schubert et donne le nombre d'intersections de deux cycles quelconques de dimensions complémentaires.*

20. **La variété des éléments $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}$ contenus dans $[n]$.** Considérons le cas d'un élément générateur $\{\alpha, \beta, \gamma\}$ composé de trois variétés planes $\{\alpha\}$, $\{\beta\}$, $\{\gamma\}$. Soit V la variété de tous les éléments $\{\alpha, \beta, \gamma\}$ de $[n]$. Considérons de nouveau une suite de variétés planes $[n-1], \cdots, [1], [0]$ telles que:

$$[n] \supset [n-1] \supset \cdots \supset [1] \supset [0]. \tag{1}$$

En prenant dans cette suite $\alpha+1$ variétés $[a_i]$, $\beta+1$ variétés $[b_j]$ et $\gamma+1$ variétés $[c_k]$, le symbole

$$\begin{bmatrix} a_0, a_1, \cdots, a_\alpha & 0 \leqq a_0 < a_1 < \cdots < a_\alpha \leqq n \\ b_0, b_1, \cdots\cdots, b_\beta & 0 \leqq b_0 < b_1 < \cdots < b_\beta \leqq n \\ c_0, c_1, \cdots\cdots, c_\gamma & 0 \leqq c_0 < c_1 < \cdots < c_\gamma \leqq n \end{bmatrix} \tag{2}$$

représentera la variété des éléments $\{\alpha, \beta, \gamma\}$ composés d'un élément $\{\alpha\}$ appartenant à $[a_0, a_1, \cdots, a_\alpha]$, d'un élément $\{\beta\}$ appartenant à $[b_0, b_1, \cdots, b_\beta]$ et d'un élément $\{\gamma\}$ appartenant à $[c_0, c_1, \cdots, c_\gamma]$. Pour que les éléments $\{\alpha\}$, $\{\beta\}$ et $\{\gamma\}$ qui composent l'élément général $\{\alpha, \beta, \gamma\}$ de la variété (2) soient des éléments généraux de leurs variétés respectives, il faut et il suffit que tout nombre a_i soit égal à un nombre b_j, où $i \leqq j$, et que tout nombre b_j soit égal à un nombre c_k, où $j \leqq k$. Lorsque ces conditions sont satisfaites, le symbole (2) sera dit irréductible et la variété qu'il définit sera appelée variété fondamentale. Cette variété fondamentale moins le lieu de ses éléments singuliers est transformée transitivement par un groupe projectif continu, un élément $\{\alpha, \beta, \gamma\}$ étant dit singulier lorsqu'un de ses éléments composants, $\{\alpha\}$, $\{\beta\}$, $\{\gamma\}$, est singulier dans sa variété respective. Les variétés fondamentales sont par suite des variétés algébriques irréductibles. Lorsqu'un symbole n'est pas irréductible, on peut le remplacer par un ou plusieurs symboles irréductibles en procédant selon les indications du paragraphe précédent.

Considérons dans V toutes les variétés fondamentales qu'on peut définir à l'aide des variétés planes de la suite (1). Elles définissent une subdivision de V en cellules algébriques. On a, en effet, la règle suivante:

Si (2) *est le symbole irréductible d'une variété fondamentale, on considère tous les symboles qu'on peut en déduire en diminuant d'une unité un des nombres a_i, b_j ou c_k. On décompose les symboles ainsi obtenus en symboles irréductibles. La variété fondamentale* (2) *devient une cellule algébrique ouverte lorsqu'on en*

enlève l'ensemble des variétés fondamentales représentées par ces symboles irréductibles.

Pour prouver la règle, il suffit de remarquer que la variété qui reste, après l'enlèvement de ces variétés-frontières, est le produit topologique d'une cellule algébrique engendrée par des éléments $\{\alpha, \beta\}$ et d'une cellule algébrique engendrée par des éléments $\{\gamma - \beta - 1\}$ d'une certaine variété $[n - \beta - 1]'$. En ce qui concerne *l'indice de Kronecker de deux variétés fondamentales de dimensions complémentaires*, on montre encore que:

$$\begin{bmatrix} a_0, \cdots, a_\alpha \\ b_0, \cdots\cdots, b_\beta \\ c_0, \cdots\cdots\cdots, c_\gamma \end{bmatrix} \cdot \begin{bmatrix} n - a_\alpha, \cdots, n - a_0 \\ n - b_\beta, \cdots\cdots\cdots, n - b_0 \\ n - c_\gamma, \cdots\cdots\cdots\cdots\cdots, n - c_0 \end{bmatrix} = 1$$

Si les symboles de deux variétés fondamentales ne se correspondent pas de cette façon, l'indice de Kronecker correspondant est nul. Donc si Γ_{2s} est un cycle sur V, on a l'homologie:

$$\Gamma_{2s} \sim \Sigma \begin{pmatrix} n - a_\alpha, \cdots, n - a_0 \\ n - b_\beta, \cdots\cdots\cdots, n - b_0 \\ n - c_\gamma, \cdots\cdots\cdots\cdots\cdots, n - c_0 \end{pmatrix} \begin{bmatrix} a_0, \cdots, a_\alpha \\ b_0, \cdots\cdots, b_\beta \\ c_0, \cdots\cdots\cdots, c_\gamma \end{bmatrix},$$

les notations étant analogues à celles des paragraphes précédents. Ceci résout en même temps le problème des caractéristiques de Schubert.

Il est bien clair qu'on a des résultats tout à fait analogues dans le cas de la variété des éléments $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}$ contenus dans $[n]$.

A titre d'exemple, donnons les bases d'homologie de la variété des éléments $\{0, 1, 2\}$ contenus dans $[3]$.

$$\begin{bmatrix} 3 \\ 2,3 \\ 1,2,3 \end{bmatrix} \qquad R_{12} = 1$$

$$\begin{bmatrix} 2 \\ 2,3 \\ 1,2,3 \end{bmatrix} \begin{bmatrix} 3 \\ 1,3 \\ 1,2,3 \end{bmatrix} \begin{bmatrix} 3 \\ 2,3 \\ 0,2,3 \end{bmatrix} \qquad R_{10} = 3$$

$$\begin{bmatrix} 1 \\ 1,3 \\ 1,2,3 \end{bmatrix} \begin{bmatrix} 2 \\ 1,2 \\ 1,2,3 \end{bmatrix} \begin{bmatrix} 2 \\ 2,3 \\ 0,2,3 \end{bmatrix} \begin{bmatrix} 3 \\ 0,3 \\ 0,2,3 \end{bmatrix} \begin{bmatrix} 3 \\ 1,3 \\ 0,1,3 \end{bmatrix} \qquad R_8 = 5$$

$$\begin{bmatrix} 0 \\ 0,3 \\ 0,2,3 \end{bmatrix} \begin{bmatrix} 1 \\ 1,2 \\ 1,2,3 \end{bmatrix} \begin{bmatrix} 1 \\ 1,3 \\ 0,1,3 \end{bmatrix} \begin{bmatrix} 2 \\ 0,2 \\ 0,2,3 \end{bmatrix} \begin{bmatrix} 3 \\ 0,3 \\ 0,1,3 \end{bmatrix} \begin{bmatrix} 2 \\ 1,2 \\ 0,1,2 \end{bmatrix} \qquad R_6 = 6$$

$$\begin{bmatrix}0\\0,2\\0,2,3\end{bmatrix}\begin{bmatrix}0\\0,3\\0,1,3\end{bmatrix}\begin{bmatrix}1\\0,1\\0,1,3\end{bmatrix}\begin{bmatrix}1\\1,2\\0,1,2\end{bmatrix}\begin{bmatrix}2\\0,2\\0,1,2\end{bmatrix} \qquad R_4 = 5$$

$$\begin{bmatrix}0\\0,1\\0,1,3\end{bmatrix}\begin{bmatrix}0\\0,2\\0,1,2\end{bmatrix}\begin{bmatrix}1\\0,1\\0,1,2\end{bmatrix} \qquad R_2 = 3$$

$$\begin{bmatrix}0\\0,1\\0,1,2\end{bmatrix} \qquad R_0 = 1$$

La somme des nombres de Betti est toujours égale à la valeur de la caractéristique d'Euler-Poincaré qui a été déterminée au §18.

21. **Remarque sur la topologie du groupe de la geométrie hermitienne elliptique.** Les variétés de Grassmann et les variétés plus générales qu'on vient d'étudier sont transformées transitivement par le groupe hermitien elliptique G et peuvent donc être considérées comme des espaces de décomposition de la variété de G. Il ne semble pas facile de subdiviser la variété de G en cellules et de déterminer de cette façon les invariants topologiques de G.[33] Cependant on arrive à une représentation assez concrète de cette variété en considérant la variété V des éléments $\{0, 1, \cdots, n-1\}$ de l'espace projectif $[n]$. Nous supposons que G est le groupe projectif de $[n]$ qui laisse invariante la forme d'Hermite $x_0\,\bar{x}_0 + x_1\,\bar{x}_1 + \cdots + x_n\,\bar{x}_n$. Nous prenons comme élément-origine de V l'élément $\{0, 1, \cdots, n-1\}_0$ composé des variétés $[\alpha]$ d'équations $x_k = 0$ pour $k > \alpha$, α étant un des nombres $0, 1, \cdots, n-1$. Le sous-groupe g qui laisse invariant l'élément-origine a pour équations:

$$x'_k = e^{i\varphi_k}\,x_k \qquad (k = 0, 1, \cdots, n). \tag{1}$$

La variété de g est homéomorphe à un tore à n dimensions (produit topologique de n cercles).

La topologie de V est connue. A chaque point de V correspond dans G une variété homéomorphe à un tore à n dimensions. A un domaine suffisamment petit de V correspond dans G le produit topologique de ce domaine par le tore à n dimensions. G n'est pas intégralement le produit topologique de V par le tore à n dimensions; car s'il en était ainsi, le groupe de Poincaré de G serait infini. Or M. E. Cartan a démontré[34] que le groupe de Poincaré de G est le groupe cyclique d'ordre $n+1$, le groupe simplement connexe $\bar{G}$ localement isomorphe à G étant le groupe linéaire unimodulaire de la forme d'Hermite $x_0\,\bar{x}_0 + \cdots + x_n\,\bar{x}_n$.

A l'aide des résultats du §2, nous pouvons vérifier que V est simplement

[33] Une formule théorique pour le calcul des nombres de Betti a été donnée par M. E. Cartan, *b*, p. 218–222.

[34] Voir E. Cartan, *d*.

connexe. A g correspond, en effet, dans $\bar{G}$ un groupe connexe $\bar{g}$ également homéomorphe à un tore à n dimensions. Il en résulte que toutes les variétés algébriques considérées jusqu'ici sont simplement connexes, car, après un choix convenable de l'élément-origine, leurs groupes d'isotropie, qui sont connexes, contiennent toujours le groupe d'isotropie g de V.

La variété V peut aussi être considérée comme un espace de décomposition du groupe $\bar{G}$, à tout point de V correspondant encore dans $\bar{G}$ un tore à n dimensions.

22. **Remarques finales.** Toutes les variétés algébriques que nous avons étudiées dans ce mémoire sont des variétés rationnelles. Chacune d'elles constitue, en effet, une cellule algébrique dont l'intérieur admet une représentation birationnelle et biunivoque sur l'espace euclidien tout entier. Il est probable que certaines des propriétés topologiques rencontrées sont des propriétés communes à toutes les variétés rationnelles. En particulier, il serait intéressant de savoir si pour toute variété rationnelle sans singularités les nombres de Betti relatifs aux dimensions impaires sont nuls et s'il n'y a pas de coefficients de torsion. On démontre facilement que *toute variété rationnelle sans singularités est simplement connexe.* D'une façon plus générale:

Deux variétés algébriques sans singularités qui se correspondent par une transformation birationnelle ont des groupes de Poincaré isomorphes.

Ceci résulte du fait que l'ensemble des points fondamentaux (c'est-à-dire des points dont chacun correspond à une infinité de points) constitue une variété dont la dimension complexe est inférieure à $d - 1$, d étant la dimension complexe des variétés données.

Il faudrait aussi étudier la topologie des *nappes réelles* des variétés de Grassmann et de leurs généralisations. On a immédiatement une subdivision de ces variétés en cellules algébriques réelles. Les variétés analogues aux variétés fondamentales introduites ici fournissent les bases pour l'homologie (mod 2). Pour certains cas il est facile de déterminer les nombres de Betti et les coefficients de torsion. Nous développerons ces résultats dans un autre article.

La méthode que nous avons employée permet d'étudier encore d'autres variétés algébriques. Elle conduit a des solutions simples et rigoureuses de certains problèmes de géométrie énumérative. Elle s'applique en particulier à la variété des coniques ou des quadriques de l'espace projectif $[n]$.